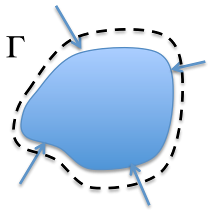
Basic Elastostatics

- The equilibrium (static) deformation of an elastic body is determined by a local balance of the effective force density:



$$0 = f_i(\underline{r}) + \nabla_j \sigma_{ij}(\underline{r})$$

External Force Field — Internal Stress Field

- For weak deformation of a linear elastic solid, we have:

$$\sigma_{ij}(\underline{r}) = 2\mu\epsilon_{ij}(\underline{r}) + \lambda\delta_{ij}\epsilon_{kk}(\underline{r})$$ Hooke's Law

$$\epsilon_{ij}(\underline{r}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ Linear Cauchy strain tensor

Basic Elastostatics

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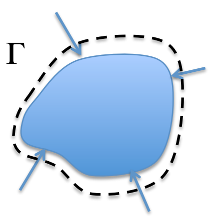


$$0 = f_i(\underline{r}) + \nabla_j \sigma_{ij}(\underline{r})$$

External Force Field — Internal Stress Field

- For weak deformation of a linear elastic solid, we have:

$$\epsilon_{ij}(\vec{r}) = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$$ Inverted-Hooke's Law

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \qquad E = \mu\frac{3\lambda+2\mu}{\lambda+\mu}$$

Navier's Equation

- Taking the gradient of the stress tensor:

$$\nabla_j \sigma_{ij} = 2\mu \nabla_j \epsilon_{ij} + \lambda \delta_{ij} \nabla_j \epsilon_{kk} = \mu \nabla^2 u_i + (\mu + \lambda) \nabla_i \nabla_j u_j$$

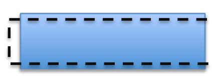
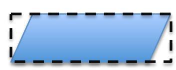
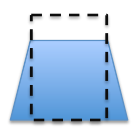
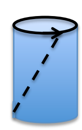
… gives Navier's equation:

$$\underline{\mathrm{f}}(\underline{\mathrm{r}}) + \mu \nabla^2 \underline{\mathrm{u}}(\underline{\mathrm{r}}) + (\lambda + \mu) \nabla \nabla \cdot \underline{\mathrm{u}}(\underline{\mathrm{r}}) = 0$$

- Boundary conditions connect surface stress to traction forces:

$$\underline{\mathrm{t}} = \underline{\sigma} \cdot \hat{n}|_{\Gamma}$$

Prototypical Deformations

- Extension/Compression: 
- Simple shear: 
- Settling under an external body force: 
- Bending: 
- Twisting: 

Dimensional Analysis

- Approximate solutions and characteristic scales can be obtained by simple dimensional analysis
- We characterize the material by a typical elastic constant E, density ρ, and linear dimension L
- External perturbations have a characteristic surface stress P or a characteristic external body force density f
- From these, we can estimate the characteristic deformation of the material

Analysis of Applied Stress

- In equilibrium, characteristic surface stress P gives rise to a more-or-less uniform level of stress $|\sigma| \sim P$ in the medium
- The linear relation between stress and strain leads to a characteristic strain level $|\varepsilon| \sim |\sigma|/E \sim P/E$ in the material
- Across the linear dimension L of the sample this leads to a variation of displacement of order: $|\Delta u| \sim L|\varepsilon| \sim LP/E$

  e.g.: a rod of hard plastic ($K \sim 100$ MPa; $L = 1$m; $A = 10\text{cm}^2$) is subjected to a compressing force of $\tau \sim 100$ N.

  Estimated length change: $|\Delta u| \sim \tau L/(AK) \sim 1$mm

Analysis of Applied Body Force

- An applied body force density on an elastic material gives rise to a stress gradients across the body due to the distributed nature of the force: $|\Delta\sigma| \sim f\, L$

- In equilibrium, this gives rise to a characteristic variation in strain across the material, $|\Delta\varepsilon| \sim |\Delta\sigma|/E \sim L/D$, where $D=E/f$ is a characteristic deformation length scale:

- Across the linear dimension L of the sample this leads to a variation of displacement of order: $|\Delta u| \sim L|\Delta\varepsilon| \sim L^2/D$

  e.g.: a cube of jello ($K \sim 1000$ Pa; $\rho=1$ g/cm^3; $L=10$cm) settles under its own weight an amount: $|\Delta u| \sim \rho g L^2/K \sim 1$cm

Saint-Venant's Principle

"The deformation due to a localized external force distribution (with vanishing total force and moment) decays rapidly on the length scale of the force distribution"

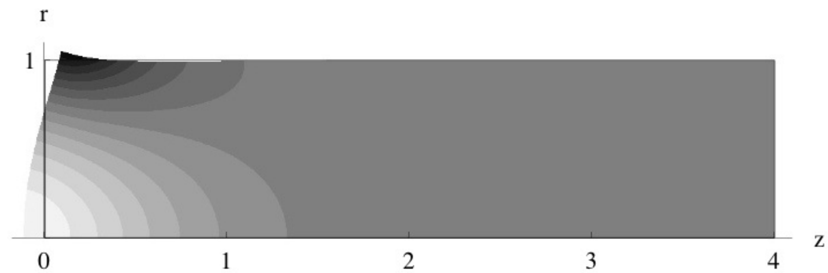
- Deformation the in the far field is unaffected by the details of the local applied force density

- Useful for approximation schemes

  c.f. the effect of dipoles in electrostatics

Saint-Venant's Illustration

- Deformation of a cylindrical rod in response to a radial surface pressure applied at one end: $P_z=P_0(2r^2-1)$
- Materials parameters: $L=4R$, $E=5P_0$, $\nu=1/3$
- Numerical finite element solution for pressure field:



N.B.: the average pressure on the end is zero,
so the extent of it's effects should be short-ranged

Saint-Venant's Illustration

- Deformation of a cylindrical rod in response to a radial surface pressure applied at one end: $P_z=P_0(2r^2-1)$
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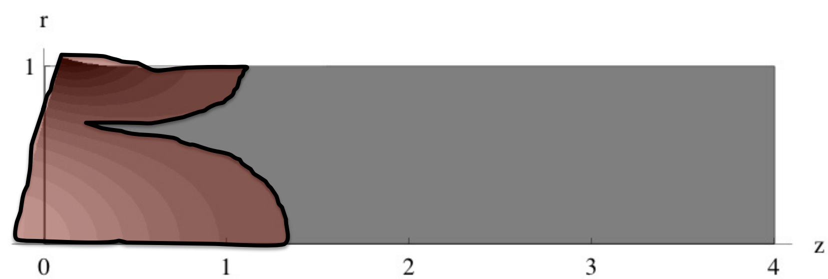


Note the localized pressure distribution at the z=0 end

Gravitational Settling

- Elastic materials will generally slump in a non-uniform manner in a constant external body force (like gravity)
- Two typical cases: constrained vs free settling:

- In the constrained case, the settling is laterally uniform

  (simple analytical solution)

- In the free case, there is shear and lateral bulging

  (no analytical solution)

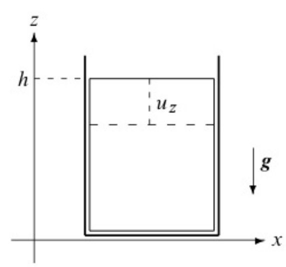
Uniform Gravitational Settling

- Strictly vertical displacement:

$$\underline{\mathrm{u}} = (0, 0, u_z(z)) \text{ with } u_z(0) = 0$$

$$\Rightarrow \epsilon_{zz} = \nabla_z u_z$$

$$(\epsilon_{i,j} = 0 \quad \text{for} \quad i, j \neq z)$$



- Hookes' law gives:

$$\sigma_{xx}(z) = \sigma_{yy}(z) = \lambda\epsilon_{zz}$$

$$\sigma_{zz}(z) = (\lambda + 2\mu)\epsilon_{zz}$$

- Cauchy's equilibrium condition with BC:

$$\nabla_z \sigma_{zz}(z) = \rho g \quad \text{with} \quad \sigma_{zz}(h) = 0$$

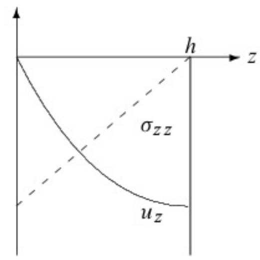
- Stress field solution: $\boxed{\sigma_{zz}(z) = -\rho g(h - z)}$

Uniform Gravitational Settling

- Pressure components:

$$P_z = -\sigma_{zz}(z) = \rho g(h - z)$$

$$P_x = P_y = P_z \lambda/(\lambda + 2\mu)$$



- Strain field solution:

$$\epsilon_{zz} = \nabla_z u_z = \frac{\sigma_{zz}(z)}{(\lambda + 2\mu)} = -\frac{(h - z)}{D}$$

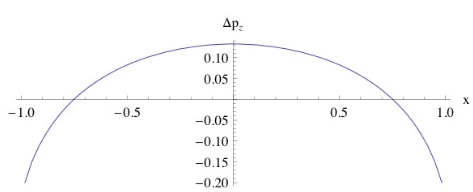
- Characteristic deformation length scale: $D = \dfrac{\lambda + 2\mu}{\rho g}$
- Displacement solution: $\nabla_z u_z = -\dfrac{(h - z)}{D}$ with $u_z(0) = 0$

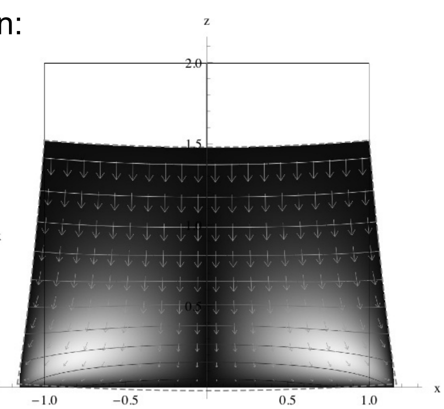
gives

$$u_z(z) = -\frac{z(2h - z)}{2D}$$

Free Gravitational Settling

- Deformation of a cylindrical rod in response to vertical gravitational field
- Materials parameters: H=2R, D=4R, ν=1/3
- Numerical finite element solution for displacement field and base pressure distribution:





Shear-Free Settling

- No-shear ansatz: $\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$
- Cauchy's equilibrium conditions become:

$$\nabla_x \sigma_{xx} = 0$$
$$\nabla_y \sigma_{yy} = 0$$
$$\nabla_z \sigma_{zz} = \rho\, g$$

σ_{xx} is independent of x
σ_{yy} is independent of y

- On free side boundaries:

$$0 = t_x = \sigma_{xx} n_x \Rightarrow \; \sigma_{xx} \equiv 0$$
$$0 = t_y = \sigma_{yy} n_y \Rightarrow \; \sigma_{yy} \equiv 0$$

- On bottom:

$$\sigma_{zz}(h) = 0 \quad \Longrightarrow \quad \boxed{\sigma_{zz}(z) = -\rho g\,(h - z)}$$

Shear-Free Settling

- Strain field solution (Inverse Hooke's Law):

$$\epsilon_{zz}(z) = -\frac{(h-z)}{D}$$
$$\epsilon_{xx}(z) = \epsilon_{xx}(z) = \nu \frac{(h-z)}{D}$$

- Characteristic deformation length scale: $D = E/\rho\, g$
- Displacement solution:

$$\nabla_z\, u_z = -(h-z)/D \;\text{ with }\; u_z(0) = 0$$
$$\nabla_x\, u_x = \nu(h-z)/D \;\text{ with }\; u_x(0) = 0$$
$$\nabla_y\, u_y = \nu(h-z)/D \;\text{ with }\; u_y(0) = 0$$

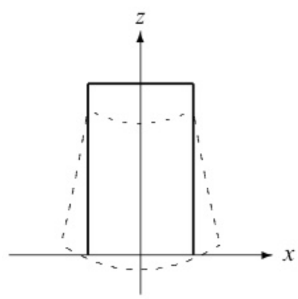
Shear-Free Settling Displacements

- Imposing shear-free conditions: $\epsilon_{xz} = \epsilon_{yz} = \epsilon_{xy} = 0$

… gives:

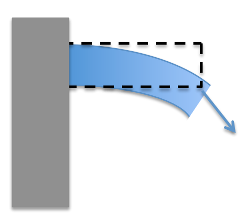
$$u_z = z(z-2h)/2D + \nu\,(x^2+y^2)/2D + K$$
$$u_x = \nu(h-z)x/D$$
$$u_y = \nu(h-z)y/D \qquad \text{with} \quad D = E/\rho\, g$$

N.B.:
- The complicated form of u_z is due to the requirement of the shear free conditions
- The condition $u_z(0)=0$ is violated by this solution For any choice of the integration constant K! In the spirit of the St. Venant principle, this doesn't affect the far field solution



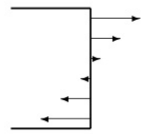
Beam Bending

- Beams are rectilinear elastic objects with uniform cross section along their length.
- They are often treated as bundles of independent elastic fibers (rays) with no inter-fiber shear stresses
- Under various applied loads, these will bend into equilibrium shapes.
- Gradual bending can be treated with linear elasticity theory

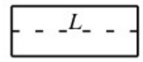


Idealized Bending

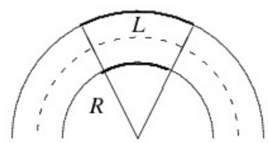
- Beam bending is *pure*:
  - only forces from surface stresses applied to ends
  - end stresses do not change the average beam length



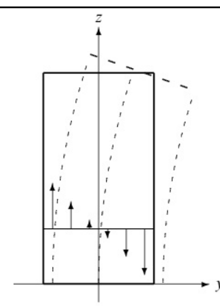
- Beam bending is *uniform*:
  - internal stresses and strains are not a function of the longitudinal coordinate along beam
- Result: all beam rays follow circular paths:



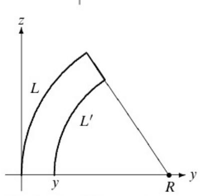
- Bending is *shear free*:
  - internal shear stress is ignored



Idealized Bending



- Consider a beam with square cross section:
  (origin at centroid of square on base of beam)
- Relation of neighboring arc lengths gives the longitudinal strain:



The length of the arc at x must satisfy $L'/(R-y) = L/R$.

$$\Rightarrow \quad \epsilon_{zz}(y) = \frac{\delta L}{L} \simeq \frac{(L' - L)}{L} \simeq \frac{-y}{R}$$

- Stresses are *shear free* and *longitudinal*:

$$\text{Hooke's Law} \quad \Rightarrow \quad \sigma_{zz} = E\epsilon_{zz}$$

$$\text{No transverse forces} \Rightarrow \quad \sigma_{xx} = \sigma_{yy} = 0$$

$$\text{No shear forces} \Rightarrow \quad \sigma_{xy} = \sigma_{yz} = \sigma_{xz} = 0$$

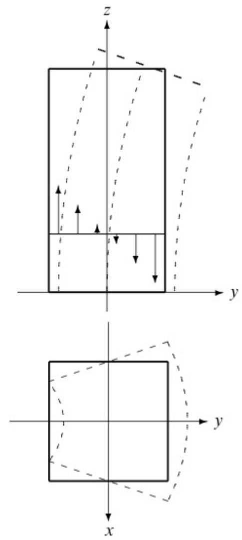
Idealized Bending

- Inverting Hooke's law gives the strains:

$$\epsilon_{xx}(y) = \epsilon_{yy}(y) = -\nu\epsilon_{zz}(y) \simeq \nu\frac{y}{R}$$

- Integration gives the displacement field using $\epsilon_{ii} = \nabla_i u_i$:

$$u_x(x,y) \simeq \nu\frac{xy}{R}$$
$$u_y(x,y,z) \simeq \frac{z^2}{2R} + \nu\frac{y^2 - x^2}{2R}$$
$$u_z(y,z) \simeq -\frac{yz}{R}$$



N.B.: form of u_y is determined by shear free conditions

Total Forces and Moments for Bending

- Idealized bending is *force free*:

$$\mathcal{F}_z = \int_A \sigma_{zz}\, dS_z = -\frac{E}{R}\int_A y\, dA = 0$$

(satisfies the conditions of St. Venant's principle)

- The moments of the longitudinal stress (bending || y):

$$\underline{d\mathcal{M}} = \underline{r} \times \underline{d\mathcal{F}} = \underline{r} \times \underline{\sigma} \cdot \underline{dS} = (\underline{r} \times \hat{z})\ \sigma_{zz}\, dA$$

$$\mathcal{M}_x = \int_A y\,\sigma_{zz}\, dS_z = -\frac{E}{R}\int_A y^2\, dA \equiv -\mathcal{M}_b \quad \underline{\text{Bending Moment}}$$

$$\mathcal{M}_y = -\int_A x\,\sigma_{zz}\, dS_z = \frac{E}{R}\int_A xy\, dA \quad (= 0 \text{ for symmetric beams})$$

$$\mathcal{M}_z = 0$$

Euler-Bernoulli Law

$$\mathcal{M}_b = \frac{EI}{R} \equiv G_b \kappa$$

κ=1/R is the curvature

G_b=E*I is the flexural rigidity

$$I = \int_A y^2 \, dA \quad \underline{\text{Area Moment}}$$

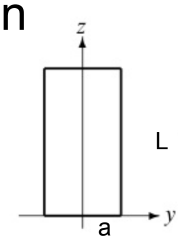
- Examples:

$$I = \frac{4}{3} a b^3 \quad \text{for rectangular beam } (x * y = 2a * 2b)$$

$$I = \frac{\pi}{4} a b^3 \quad \text{for elliptical beam } (x * y = 2a * 2b)$$

$$I = \frac{\pi}{4} (b^4 - a^4) \quad \text{for circular pipe (radii } a < b)$$

Idealized Bending vs Extension



- Consider a long beam with length L and cross section A=a^2:
- A longitudinal force F_z=F gives a longitudinal displacement:

$$u_z \simeq \frac{LF}{AE}$$

- A transverse force F_y=F gives a transverse displacement:

$$u_y \simeq \frac{L^2}{2R} \sim \frac{L^3 F}{EI} \quad \text{since } 1/R = \frac{\mathcal{M}_b}{EI} \sim \frac{LF}{EI}$$

- Ratio of longitudinal to transverse displacement is small:

$$\left| \frac{u_z}{u_y} \right| \simeq \frac{I}{AL^2} \sim \left(\frac{a}{L} \right)^2 \ll 1 \quad \text{since } I \sim Aa^2$$

Bending Energy

- Consider a long beam with length L and cross section A bent ideally with:

$$\epsilon_{xx}(y) = \epsilon_{yy}(y) = -\nu \epsilon_{zz}(y) \simeq \nu \frac{y}{R}$$
$$\sigma_{zz} = E\epsilon_{zz}$$

- The energy density for bending is:

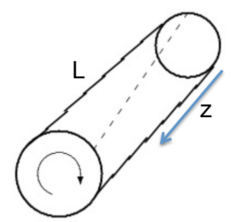
$$e_b = \frac{1}{2}\underline{\sigma} : \underline{\epsilon} = \frac{1}{2}\sigma_{zz}\,\epsilon_{zz} = \frac{1}{2}E\,\epsilon_{zz}^2 \simeq \frac{Ey^2}{2R^2}$$

- This gives a bending energy per unit length:

$$\frac{\partial \mathcal{E}_b}{\partial \ell} = \frac{\mathcal{E}_b}{L} = \frac{1}{L}\int_V e_b\,dV \simeq \frac{E}{2R^2}\int_A y^2\,dA \equiv \frac{EI}{2R^2} = \frac{\mathcal{M}_b^2}{2EI}$$

Idealized Twisting

- Consider a long circular beam with length L and radius a:

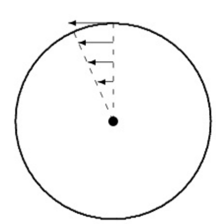


- A pure torsion rotates the beam by small, uniform amount τ per unit length (τL<<1):

Pure torsion consists of rotating every cross section by a fixed amount per unit of length.

- At a position z, the accumulated twist is given by the axial vector angle: $\underline{\phi} = \tau\, z\, \hat{e}_z$

- The local displacement field is given by:



$$\underline{u} = \underline{\phi} \times \underline{r} = \tau\, z\, \hat{e}_z \times \underline{r} = \tau\, z(-y, x, 0)$$

(a purely tangential displacement)

Twisting Strains and Stresses

- The gradient tensor is: $$\underline{\nabla u} = \begin{pmatrix} 0 & -\tau z & -\tau y \\ \tau z & 0 & \tau x \\ 0 & 0 & 0 \end{pmatrix}$$
- The non-trivial strain tensor components are:

$$\epsilon_{xz} = \epsilon_{zx} = -\frac{1}{2}\tau y \qquad \epsilon_{yz} = \epsilon_{zy} = +\frac{1}{2}\tau x$$

- Hooke's Law with vanishing diagonal strains gives exclusively shear components in the stress tensor:

$$\sigma_{xz} = \sigma_{zx} = -\mu\tau\, y \qquad \sigma_{yz} = \sigma_{zy} = +\mu\tau\, x$$

- These are consistent with the boundary conditions:

$$\underline{\sigma} \cdot \hat{e}_r\big|_{r=a} = 0$$

$$\underline{\sigma} \cdot \hat{e}_z\big|_{z=0,L} = \mu\tau\hat{e}_z \times \underline{r}\big|_{z=0,L}$$

Twisting Moment

- Total moment of force around twisting axis (z):

$$\underline{d\mathcal{M}} = \underline{r} \times \underline{d\mathcal{F}} = \underline{r} \times \underline{\sigma} \cdot \underline{dS} = (x\sigma_{yz} - y\sigma_{xz})\ \hat{z}\, dA$$

gives:

$$\mathcal{M}_t = \int_A (x\,\sigma_{yz} - y\,\sigma_{xz})\, dA = \mu\tau \int_A (x^2 + y^2)\, dA$$

- Euler-Bernoulli analog for twist:

$$\boxed{\mathcal{M}_t = \mu J \tau \equiv G_t\, \tau}$$

G_t=μ*J is the torsional rigidity

$$J = \int_A (x^2 + y^2)\, dA$$

e.g.: $J = \frac{\pi}{2}(b^4 - a^4)$ for circular pipe (radii $a < b$)

Twisting Energy

- Consider a long beam with length L and cross section A twisted ideally with:

$$\epsilon_{xz} = \epsilon_{zx} = -\frac{1}{2}\tau y \qquad \sigma_{xz} = \sigma_{zx} = -\mu\tau\, y$$

$$\epsilon_{yz} = \epsilon_{zy} = +\frac{1}{2}\tau x \qquad \sigma_{yz} = \sigma_{zy} = +\mu\tau\, x$$

- The energy density for bending is:

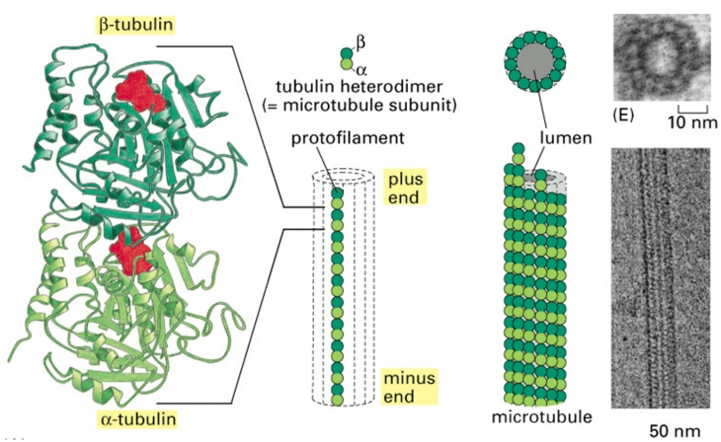
$$e_t = \frac{1}{2}\underline{\sigma} : \underline{\epsilon} = \frac{1}{2}\, 2\left(\sigma_{xz}\,\epsilon_{xz} + \sigma_{yz}\,\epsilon_{yz}\right) = \frac{1}{2}\mu\,\tau^2\left(x^2 + y^2\right)$$

- This gives a twisting energy per unit length:

$$\frac{\partial \mathcal{E}_t}{\partial \ell} = \frac{\mathcal{E}_t}{L} = \frac{1}{L}\int_V e_t\, dV \simeq \frac{1}{2}\mu\,\tau^2 \int_A (x^2 + y^2)\, dA \equiv \frac{1}{2}\mu\tau^2 J = \frac{\mathcal{M}_t^2}{2\mu\, J}$$

Microtubules

- Protofiliments form from oriented dimers of tubulin
- 13 staggered protofiliments form the hollow tubule
- Directional assembly and disassembly



Microtubule Mechanics

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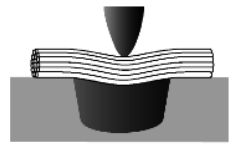
Nanomechanics of Microtubules

A. Kis,[1,*] S. Kasas,[2,3] B. Babić,[4] A. J. Kulik,[1] W. Benoît,[1] G. A. D. Briggs,[1,†] C. Schönenberger,[4] S. Catsicas,[2] and L. Forró[1]
[1]Institute of Physics of Complex Matter, EPFL, CH-1015 Lausanne, Switzerland
[2]Institut de Neurosciences, EPFL, CH-1015 Lausanne, Switzerland
[3]Institut de Biologie Cellulaire et de Morphologie, UNIL, CH-1005 Lausanne, Switzerland
[4]Institute of Physics, University of Basel, CH-4056 Basel, Switzerland

- AFM study of tubule response to loading:

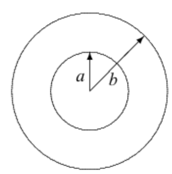


- Shear modulus is ~100 times smaller than Young's:

  $\mu \approx 1.4$ MPa $E \approx 100$ Mpa

Continuum Microtubule Model

- Isotropic elastic cylindrical shell model:



a≈ 15 nm and b≈ 25 nm give:

$$I = \frac{\pi}{4}(b^4 - a^4) \simeq 2.7 \times 10^{-31}\,\mathrm{m}^4$$

$$J = 2I = \simeq 5.4 \times 10^{-31}\,\mathrm{m}^4$$

- Young's and shear moduli from experiment:

  $\mu \approx 1.4$ MPa $E \approx 100$ Mpa

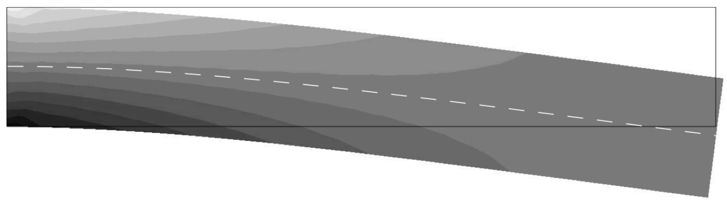
- We can estimate the flexural and torsional rigidity

$$G_b = E\,I \simeq 2.7 \times 10^{-23}\,\mathrm{N\,m}$$

$$G_t = \mu\,J \simeq 7.5 \times 10^{-25}\,\mathrm{N\,m}$$

Rod Bending Example

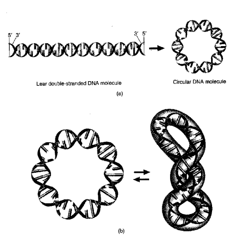
- Deformation of a cantilever by its own weight (Fig 10.1):
- Numerical solution for longitudinal stress field σ_{zz}:

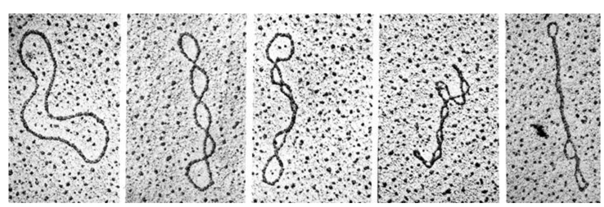


- Dashed line is slender beam theory for the centroid

Super-Twisted DNA Example

- Closed DNA loop conformation is a combination of bending, twisting, and writhing:





- The relative amount of twisting and writhing is a topological invariant: $Tw + Wr = \text{Const}$

Twisting number: Tw= # of internal twists in the DNA strand
Writhing number: Wr= # of external twists in the DNA strand