Assignment 3
Continuous Matter 4335/8191(B)

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1 Harden, Problem 1)

Without assuming small deformations, find the Cauchy strain tensor resulting from the following deformation: $\vec{r} \to \vec{r}' = \vec{f}(\vec{r})$ **where**

$$\vec{f} = (\alpha x + \lambda\beta y)\,\hat{x} + \beta y\hat{y} + \gamma z\hat{z}.$$

The displacement field is

$$\begin{aligned}\vec{u} &= \vec{f} - \vec{r} \\ &= ((\alpha-1)\,x + \lambda\beta y)\,\hat{x} + (\beta-1)\,y\hat{y} + (\gamma-1)\,z\hat{z}\end{aligned} \tag{1}$$

and we know that the (Eulerian) strain is

$$\varepsilon_{ij} = \frac{1}{2}\left(\nabla_i u_i + \nabla_j u_i + \underbrace{\sum_k \nabla_i u_k \nabla_j u_k}_{\text{neglected for small deformations}}\right) \tag{2}$$

so to begin let's calculate all needed $\nabla_i u_j$ terms.

$$\begin{array}{ccc} \frac{\partial u_x}{\partial x} = \alpha - 1 & \frac{\partial u_x}{\partial y} = \lambda\beta & \frac{\partial u_x}{\partial z} = 0 \\ \frac{\partial u_y}{\partial x} = 0 & \frac{\partial u_y}{\partial y} = \beta - 1 & \frac{\partial u_y}{\partial z} = 0 \\ \frac{\partial u_z}{\partial x} = 0 & \frac{\partial u_z}{\partial y} = 0 & \frac{\partial u_z}{\partial z} = \gamma - 1 \end{array}. \tag{3}$$

We can immediately feed Eq. (3) into Eq. (2) to get our solution (which recall is symmetric):

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{2}\left\{(\alpha-1) + (\alpha-1) + \left[(\alpha-1)^2 + 0 + 0\right]\right\} = \frac{\alpha^2 - 1}{2} \\
\varepsilon_{yy} &= \frac{1}{2}\left\{(\beta-1) + (\beta-1) + \left[(\lambda\beta)^2 + (\beta-1)^2 + 0\right]\right\} = \frac{\left(\lambda^2+1\right)\beta^2 - 1}{2} \\
\varepsilon_{zz} &= \frac{1}{2}\left\{(\gamma-1) + (\gamma-1) + \left[0 + 0 + (\gamma-1)^2\right]\right\} = \frac{\gamma^2 - 1}{2} \\
\varepsilon_{xy} = \varepsilon_{yx} &= \frac{1}{2}\left\{0 + \lambda\beta + \left[(\alpha-1)\,\lambda\beta + 0 + 0\right]\right\} = \frac{\alpha\lambda\beta}{2} \\
\varepsilon_{zx} = \varepsilon_{xz} &= \frac{1}{2}\left\{0 + 0 + [0+0+0]\right\} = 0 \\
\varepsilon_{yz} = \varepsilon_{zy} &= \frac{1}{2}\left\{0 + 0 + [0+0+0]\right\} = 0
\end{aligned}$$

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2}\begin{bmatrix} \alpha^2 - 1 & \alpha\lambda\beta & 0 \\ \alpha\lambda\beta & \left(\lambda^2+1\right)\beta^2 - 1 & 0 \\ 0 & 0 & \gamma^2 - 1 \end{bmatrix}}. \tag{4}$$

If the opposite sign had been chosen (Lagrangian)

$$\varepsilon_{ij} = \frac{1}{2}\left(\nabla_i u_i + \nabla_j u_i - \sum_k \nabla_i u_k \nabla_j u_k\right) \tag{5}$$

then

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{2}\left\{(\alpha-1)+(\alpha-1)-\left[(\alpha-1)^2+0+0\right]\right\} = \frac{2\alpha-2-\alpha^2+2\alpha-1}{2} \\
\varepsilon_{yy} &= \frac{1}{2}\left\{(\beta-1)+(\beta-1)-\left[(\lambda\beta)^2+(\beta-1)^2+0\right]\right\} = \frac{2\beta-2-\lambda^2\beta^2-\beta^2+2\beta-1}{2} \\
\varepsilon_{zz} &= \frac{1}{2}\left\{(\gamma-1)+(\gamma-1)-\left[0+0+(\gamma-1)^2\right]\right\} = \frac{2\gamma-2-\gamma^2+2\gamma-1}{2} \\
\varepsilon_{xy} = \varepsilon_{yx} &= \frac{1}{2}\left\{0+\lambda\beta-\left[(\alpha-1)\lambda\beta+0+0\right]\right\} = \frac{2\lambda\beta-\alpha\lambda\beta}{2} \\
\varepsilon_{zx} = \varepsilon_{xz} &= \frac{1}{2}\left\{0+0-\left[0+0+0\right]\right\} = 0 \\
\varepsilon_{yz} = \varepsilon_{zy} &= \frac{1}{2}\left\{0+0-\left[0+0+0\right]\right\} = 0
\end{aligned}$$

$$\underline{\underline{\varepsilon}} = -\frac{1}{2}\begin{bmatrix} \alpha^2-4\alpha+3 & \lambda\beta(\alpha-2) & 0 \\ \lambda\beta(\alpha-2) & \left(\lambda^2+1\right)\beta^2-4\beta+3 & 0 \\ 0 & 0 & \gamma^2-4\gamma+3 \end{bmatrix}. \tag{6}$$

2 Harden, Problem 2

The stress field on a spherically symmetric body is

$$\sigma_{rr} = -\left(A+\frac{B}{r^3}\right) \tag{7}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = -\left(A+\frac{C}{r^3}\right). \tag{8}$$

with A, B, C constant. What are the conditions that A, B, C must obey for equilibrium in the absence of an external force?

We have to be a little careful here because

$$\vec{f} + \vec{\nabla}\cdot\underline{\underline{\sigma}} = 0 \tag{9}$$

i.e. we are dealing with the divergence of a tensor not a vector so there are terms in each component due to the change of the unit vectors.

The symmetry of the problem tells us not to concern ourselves with the $\hat{\theta}$ or $\hat{\phi}$ components. So dealing only with the $\hat{r}$ component we have

$$\begin{aligned}
0 &= \hat{r}\cdot\left(\vec{\nabla}\cdot\underline{\underline{\sigma}}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin(\theta\sigma_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\sigma_{\phi r} - \frac{\sigma_{\theta\theta}+\sigma_{\phi\phi}}{r} \\
&= \left[\frac{\partial\sigma_{rr}}{\partial r}+\frac{2\sigma_{rr}}{r}\right] + 0 + 0 - \frac{2\sigma_{\theta\theta}}{r} \\
&= -\frac{\partial}{\partial r}\left(A+\frac{B}{r^3}\right) - \frac{2}{r}\left(A+\frac{B}{r^3}\right) + \frac{2}{r}\left(A+\frac{C}{r^3}\right) \\
&= \frac{3B}{r^4} - \frac{2B}{r^4} + \frac{2C}{r^4}
\end{aligned}$$

$$\boxed{B+2C=0}. \tag{10}$$

A and C would be determined from some specific set boundary conditions.

3 Harden, Problem 3

The stress in a body is

$$\underline{\underline{\sigma}} = a \begin{bmatrix} x^2 y & \left(b^2 - y^2\right) x & 0 \\ \left(b^2 - y^2\right) x & \left(y^2 - 3b^2\right) y/3 & 0 \\ 0 & 0 & 2bz^2 \end{bmatrix} \tag{11}$$

where a and b are constants. Determine the force density required for the body to be in equilibrium.

All we have to do is balance the forces:

$$\begin{aligned} \vec{f} + \vec{\nabla} \cdot \underline{\underline{\sigma}} &= 0 \\ \vec{f} &= -\vec{\nabla} \cdot \underline{\underline{\sigma}} \\ &= - \begin{bmatrix} \nabla_x \sigma_{xx} + \nabla_y \sigma_{xy} + \nabla_z \sigma_{xz} \\ \nabla_x \sigma_{yx} + \nabla_y \sigma_{yy} + \nabla_z \sigma_{yz} \\ \nabla_x \sigma_{zx} + \nabla_y \sigma_{zy} + \nabla_z \sigma_{zz} \end{bmatrix} \\ &= - \begin{bmatrix} 2axy - 2axy + 0 \\ \left(ab^2 - ay^2\right) + \left(\frac{3ay^2}{3} - ab^2\right) + 0 \\ 0 + 0 + 4bz \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ 4bz \end{bmatrix} \end{aligned}$$

$$\boxed{\vec{f} = -4bz\hat{z}}. \tag{12}$$

4 Harden, Problem 4

A cylindrical solid of radius a and height h is placed between two rigid plates. The top plate is displaced downward an amount δ.

4.1 Part a)

Write the boundary conditions for if the cylinder is bonded to the top and bottom plate. Bonding the cylinder to the plate ensures that at each plate there is no $\hat{r}$ or $\hat{\theta}$ displacements at the plates. At the bottom plate ($z = 0$) there is no displacement:

$$\boxed{u_r\left(z = 0, r\right) = 0; \qquad u_\theta\left(z = 0, r\right) = 0; \qquad u_z\left(z = 0, r\right) = 0}. \tag{13}$$

At the top plate ($z = h$) there is only displacement in the $\hat{z}$ direction

$$\boxed{u_r\left(z = h, r\right) = 0; \qquad u_\theta\left(z = h, r\right) = 0; \qquad u_z\left(z = h, r\right) = -\delta}. \tag{14}$$

As a side note, notice that from the symmetry of the problem there should be no θ dependence in the stress so we can also say right away that

$$\sigma_{r\theta}\left(z, r\right) = 0; \qquad \sigma_{z\theta}\left(z, r\right) = 0; \qquad \sigma_{\theta\theta}\left(z, r\right) = 0. \tag{15}$$

4.2 Part b)

Write the boundary conditions for if the plates are frictionless. The $\hat{z}$ component of the displacement must be the same as above

$$\boxed{u_z\left(z = 0, r\right) = 0; \qquad u_z\left(z = h, r\right) = -\delta} \tag{16}$$

and we still have the same symmetry

$$\sigma_{r\theta}\left(z, r\right) = 0; \qquad \sigma_{z\theta}\left(z, r\right) = 0; \qquad \sigma_{\theta\theta}\left(z, r\right) = 0. \tag{17}$$

There is no traction at the plates so

$$\boxed{\sigma_{rz}\left(z=0,r\right)=0; \qquad \sigma_{rz}\left(z=h,r\right)=0} \tag{18}$$

and there can be no stress on the surface of the cylinder $(r=a)$

$$\boxed{\sigma_{rr}\left(z,r=a\right)=0; \qquad \sigma_{rz}\left(z,r=a\right)=0} \tag{19}$$

5 Harden, Problem 5

An external hydrostatic pressure (magnitude P) is applied to the surface of a spherical body of radius b with a concentric rigid spherical inclusion of radius a.

5.1 Part a)

Determine the displacement and stress fields in the spherical body.

The displacement obeys the NAvier-Cauchy equilibrium condition

$$0=\vec{f}+\mu\nabla^2\vec{u}+(\lambda+\mu)\,\vec{\nabla}\vec{\nabla}\cdot\vec{u}$$

Without body forces and considering the spherical symmetry this becomes

$$0=\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)\right], \tag{20}$$

which generally has the solution

$$\boxed{u_r=Ar+\frac{B}{r^2}}. \tag{21}$$

To find the integration constants A and B we next consider the strain

$$\begin{aligned}
\varepsilon_{rr}&=\nabla_r u_r=\frac{\partial u_r}{\partial r}=A-\frac{2B}{r^3}\\
\varepsilon_{\theta\theta}&=\frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)=A+\frac{B}{r^3}\\
\varepsilon_{\phi\phi}&=\frac{1}{r\sin\theta}\left(\frac{\partial u_\phi}{\partial\phi}+u_r\sin\theta+u_\theta\cos\theta\right)=\frac{u_r\sin\theta}{r\sin\theta}=\frac{u_r}{r}=A+\frac{B}{r^3}.
\end{aligned}$$

Now we're ready to find the stress through Hooke's law:

$$\sigma_{ij}=2\mu\varepsilon_{ij}+\lambda\delta_{ij}\sum_k\varepsilon_{kk} \tag{22}$$

$$\begin{aligned}
\sigma_{rr}&=2\mu\varepsilon_{rr}+\lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}\right)\\
&=2\mu\left(A-\frac{2B}{r^3}\right)+\lambda\left(A-\frac{2B}{r^3}+2A+\frac{2B}{r^3}\right)\\
&=\boxed{\left(2\mu+3\lambda\right)A-\frac{4\mu}{r^3}B}
\end{aligned} \tag{23}$$

$$\begin{aligned}
\sigma_{\theta\theta}=\sigma_{\phi\phi}&=2\mu\varepsilon_{\theta\theta}+\lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}\right)\\
&=2\mu\left(A+\frac{B}{r^3}\right)+\lambda\left(3A\right)\\
&=\boxed{\left(2\mu+3\lambda\right)A+\frac{2\mu}{r^3}B}.
\end{aligned} \tag{24}$$

We are now ready to find the constants A and B from the boundary conditions

1. At the core $r = a$ the displacement must be equal to zero

$$u_r(r = a) = 0 \tag{25}$$

2. At the surface $(r = b)$ the stress must be equal to the hydrostatic pressure

$$\sigma_{rr}(r = b) = -P \tag{26}$$

From Eq. (25) in Eq. (21), we see that A and B are related

$$\begin{aligned} u_r(r = a) &= Aa + \frac{B}{a^2} = 0 \\ A &= -\frac{B}{a^3}. \end{aligned} \tag{27}$$

From Eq. (26) in Eq. (23), we see

$$\begin{aligned} \sigma_{rr}(r = b) &= (2\mu + 3\lambda) A - \frac{4\mu}{b^3} B = -P \\ -P &= (2\mu + 3\lambda) A + \frac{4\mu A a^3}{b^3} \end{aligned}$$

$$\boxed{A = \frac{-P}{2\mu + 3\lambda + 4\mu\frac{a^3}{b^3}}} \tag{28}$$

$$\boxed{B = \frac{Pa^3}{2\mu + 3\lambda + 4\mu\frac{a^3}{b^3}}}, \tag{29}$$

which we can substitute into our stress and displacement fields

$$\boxed{u_r(r) = \frac{-Pr}{2\mu + 3\lambda + 4\mu\frac{a^3}{b^3}}\left(1 - \frac{a^3}{r^3}\right)} \tag{30}$$

$$\boxed{\sigma_{rr} = -P\frac{2\mu + 3\lambda + 4\mu\frac{a^3}{r^3}}{2\mu + 3\lambda + 4\mu\frac{a^3}{b^3}}} \tag{31}$$

$$\boxed{\sigma_{\theta\theta} = \sigma\phi\phi = -P\frac{2\mu + 3\lambda - 2\mu\frac{a^3}{r^3}}{2\mu + 3\lambda + 4\mu\frac{a^3}{b^3}}} \tag{32}$$

5.2 Part b)

Use Part a) to estimate the stress at the surface of the rigid inclusion in an infinite elastic medium.

We take $b \to \infty$ (a stays finite).

$$u_r(r) = \frac{-Pr}{2\mu + 3\lambda}\left(1 - \frac{a^3}{r^3}\right) \tag{33}$$

$$\begin{aligned} \sigma_{rr} &= -P\frac{2\mu + 3\lambda + 4\mu\frac{a^3}{r^3}}{2\mu + 3\lambda} \\ &= -P\left[1 + \frac{4\mu}{2\mu + 3\lambda}\frac{a^3}{r^3}\right] \end{aligned} \tag{34}$$

$$\begin{aligned} \sigma_{\theta\theta} = \sigma\phi\phi &= -P\frac{2\mu + 3\lambda - 2\mu\frac{a^3}{r^3}}{2\mu + 3\lambda} \\ &= -P\left[1 - \frac{2\mu}{2\mu + 3\lambda}\frac{a^3}{r^3}\right]. \end{aligned} \tag{35}$$

So then at the surface of the rigid inclusion we have

$$\boxed{u_r\left(r=a\right)=0} \tag{36}$$

$$\boxed{\sigma_{rr}\left(r=a\right)=-P\left[1+\frac{4\mu}{2\mu+3\lambda}\right]} \tag{37}$$

$$\boxed{\sigma_{\theta\theta}\left(r=a\right)=-P\left[1-\frac{2\mu}{2\mu+3\lambda}\right]} \tag{38}$$