Assignment 2
Continuous Matter 4335/8191(B)

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1 Problem 1)

Given the stream function $\psi = x - x^3 y/2$, determine if

- **mass is conserved**
- **the flow is irrotational and**
- **the velocity potential.**

For each part we require the velocity field:

$$v_x = \frac{\partial \psi}{\partial y} = -\frac{x^3}{2} \tag{1}$$

$$v_y = -\frac{\partial \psi}{\partial x} = -\left(1 - \frac{3}{2}x^2 y\right) = \frac{3}{2}x^2 y - 1. \tag{2}$$

1.1 Part a)

We can not simply assume that if density ρ is constant that mass is conserved. We must consider the continuity equation (assuming that density is constant) for this potentially non-physical flow:

$$\begin{aligned} \vec{\nabla} \cdot \vec{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \\ &= \left(-\frac{3x^2}{2}\right) + \left(\frac{3x^2}{2} - 0\right) \end{aligned}$$

$$\boxed{\vec{\nabla} \cdot \vec{v} = 0}. \tag{3}$$

Therefore, mass is conserved.

1.2 Part b)

The flow is irrotational if the vorticity is zero.

$$\begin{aligned} \vec{\omega} &= \vec{\nabla} \times \vec{v} \\ &= \left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right] \hat{x} + \left[\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right] \hat{y} + \left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right] \hat{z} \\ &= [0]\, \hat{x} + [0]\, \hat{y} + [3xy - 0]\, \hat{z} \end{aligned}$$

$$\boxed{\vec{\omega} = 3xy\hat{z}}. \tag{4}$$

So the flow is not irrotational.

1.3 Part c)

Because the vorticity is non-zero we will be unable to define a velocity potential ϕ. To see this explicitly, we try to integrate the velocity components and see that the potentials found from each component are incompatible. First the x-component:

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} = v_x \\ \phi &= C_x(y) + \int v_x dx \\ &= C_x(y) + \int \left(-\frac{x^3}{2}\right) dx\end{aligned}$$

$$\boxed{\phi = C_x(y) - \frac{x^4}{8}}. \tag{5}$$

The integration constant C_x is constant for variations in x but can indeed vary as a function of y. Now we consider the y-component and see that it can not be compatibly with Eq. (5):

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x} = v_y \\ \phi &= C_y(x) + \int v_y dy \\ &= C_y(x) + \int \left(\frac{3}{2}x^2 y - 1\right) dy\end{aligned}$$

$$\boxed{\phi = C_y(x) + \frac{3}{4}x^2 y^2 - y}. \tag{6}$$

We now attempt to compare Eq. (5) and Eq. (6) to determine the functions $C_y(x)$ and $C_y(x)$. We quickly see that since Eq. (6) has a term that depends on both x and y, it can never be made compatible with Eq. (5), which only allows a function of y alone and so we have verified that the velocity potential is ill-defined just as we knew from Part b)

2 Problem 2)

The flow field of resulting from the superposition of a doubled (source-sink pair with vanishingly small separation – or equivalently viewed from sufficiently far away) superimposed with rectilinear flow is

$$u_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta \tag{7}$$

$$u_\theta = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta. \tag{8}$$

2.1 Part a)

Calculate the velocity potential ϕ.

We start with the form of the velocity potential from the radial component first:

$$
\begin{aligned}
u_r &= \frac{\partial \phi}{\partial r} \\
\phi &= C_r(\theta) + \int u_r dr \\
&= C_r(\theta) + \int U\left(1 - \frac{R^2}{r^2}\right)\cos\theta dr \\
&= C_r(\theta) + U\cos\theta \int \left(1 - \frac{R^2}{r^2}\right) dr
\end{aligned}
$$

$$
\boxed{\phi = C_r(\theta) + U\left(r + \frac{R^2}{r}\right)\cos\theta}. \tag{9}
$$

Likewise, we consider the angular component:

$$
\begin{aligned}
u_\theta &= -\frac{1}{r}\frac{\partial \phi}{\partial \theta} \\
\phi &= C_\theta(r) - \int r u_\theta d\theta \\
&= C_\theta(r) + \int rU\left(1 + \frac{R^2}{r^2}\right)\sin\theta d\theta \\
&= C_\theta(r) + U\left(r + \frac{R^2}{r}\right)\int \sin\theta d\theta
\end{aligned}
$$

$$
\boxed{\phi = C_\theta(r) + U\left(r + \frac{R^2}{r}\right)\cos\theta}. \tag{10}
$$

Comparing the two forms of ϕ we see that we must have $C_r(\theta) = C_\theta(r) = C$ where C can only be a numerical constant:

$$
\boxed{\phi = C + U\left(r + \frac{R^2}{r}\right)\cos\theta}. \tag{11}
$$

2.2 Part b)

What other type of system does this velocity potential correspond to?

This is identical to an inviscid profile of a cylindrical obstacle subject to a uniform flow.

3 Problem 3)

Oil flows down a vertical plate. Consider fully formed flow (*i.e.* independent of z) with a constant thickness δ. Please see the problem sheet for a diagram and definitions.

Ideally, the velocity profile w can only be a function of x and the atmosphere offers no shear resistance. In the lectures we looked at flow down an inclined plane. This is just a specific case with $\theta = \pi/2$.

For completeness sake, we go over the derivation really quickly. It's fully developed with no time dependence and low inertia. There is no pressure gradient either, which leads to the Navier-Stokes equation reduces to

$$
\begin{aligned}
\eta\nabla^2 v &= -\vec{f} \\
&= -\rho\vec{g} \\
&= -\rho g\sin\theta.
\end{aligned}
$$

x	g	δ	η	ρ	v_x
L	LT^{-2}	L	$MT^{-1}L^{-1}$	ML^{-3}	LT^{-1}

Table 1: Parameters and their units.

Our coordinate system is the standard for inclined planes that we all used so many times in first year. We integrate to find the general solution:

$$v_z = a + bx - \frac{\rho g \sin\theta}{2\eta} x^2. \tag{12}$$

The boundary equations are

No-slip

$$v_z\,(x = 0) = 0, \tag{13}$$

which requires that $a = 0$.

No-shear

$$\left.\frac{\partial v_z}{\partial y}\right|_{x=\delta} = 0, \tag{14}$$

which demands that $b = \delta\rho g \sin\theta/\eta$.

Therefore, the velocity profile is

$$\boxed{v_z = \left(\frac{\rho g \sin\theta}{2\eta}\right)\left[2\delta x - x^2\right]} \tag{15}$$

such that when $\theta = \pi/2$, the velocity profile is

$$\boxed{v_z = \left(\frac{\rho g}{2\eta}\right)\left[2\delta x - x^2\right]}. \tag{16}$$

3.1 Part b)

Given the film thickness and the slope of the velocity at the wall, what is the viscosity? The slope at the wall is

$$\begin{aligned}\left.\frac{\partial v_z}{\partial y}\right|_{x=0} &= \frac{g}{2\eta}\left[2\delta - 2x\right]_{x=0} \\ &= \frac{g\delta}{\eta}\end{aligned}$$

$$\boxed{\eta = \frac{\rho g \delta}{\left[\partial v_z/\partial y\right]_{x=0}}}. \tag{17}$$

3.2 Part c)

Use the Π-theorem to rewrite the velocity profile in terms of dimensionless parameters.

The parameters and their units are shown in Table 1. You can choose any set of three to focus on. In my mind some are more obvious than others (both x and δ are lengths so it would be weird to chose both – that being said a couple students did and that works equally well). Here I'm going to show what happens if we choose the set $\{x, v_x, \eta\}$. This choice means we will always use the combination $g^a\delta^b\rho^c$ to find the non-dimensionalization.

Dimensionless distance:

$$\begin{aligned}\Pi_1 &= x^1 g^a \delta^b \rho^c \\ &\sim [x]^1 [g]^a [\delta]^b [\rho]^c \\ &\sim L^1 L^a T^{-2a} L^b M^c L^{-3c}\end{aligned}$$

- T: $-2a = 0 \rightarrow a = 0$
- M: $c = 0$
- L: $1 + a + b - 3c = 0 \rightarrow b = -1$

Therefore

$$\boxed{\Pi_1 = \frac{x}{\delta} \equiv \tilde{x}}, \tag{18}$$

where we have called $\Pi_1 \equiv \tilde{x}$ to emphasize that it is a dimenionless length.

Dimensionless velocity:

$$\begin{aligned}\Pi_2 &= v_x^1 g^a \delta^b \rho^c \\ &\sim [v_x]^1 [g]^a [\delta]^b [\rho]^c \\ &\sim L^1 T^{-1} L^a T^{-2a} L^b M^c L^{-3c}\end{aligned}$$

- M: $c = 0$
- T: $-1 - 2a = 0 \rightarrow a = -1/2$
- L: $1 + a + b - 3c = 0 \rightarrow b = -1/2$

Therefore

$$\boxed{\Pi_2 = \frac{v_x}{\sqrt{\delta g}} \equiv \tilde{v}}, \tag{19}$$

where we have called $\Pi_2 \equiv \tilde{v}$ to emphasize that it is a dimenionless velocity.

Dimensionless viscosity:

$$\begin{aligned}\Pi_3 &= \eta^1 g^a \delta^b \rho^c \\ &\sim [\eta]^1 [g]^a [\delta]^b [\rho]^c \\ &\sim M^1 T^{-1} L^{-1} L^a T^{-2a} L^b M^c L^{-3c}\end{aligned}$$

- T: $-1 - 2a = 0 \rightarrow a = -1/2$
- M: $1 + c = -1$
- L: $-1 + a + b - 3c = 0 \rightarrow b = -3/2$

Therefore

$$\boxed{\Pi_3 = \frac{2\eta}{\rho \delta^{3/2} g^{1/2}} = \frac{2\nu}{\delta^{3/2} g^{1/2}} \equiv \tilde{\eta}}, \tag{20}$$

where we have called $\Pi_3 \equiv \tilde{\eta}$ to emphasize that it is a dimenionless viscosity and we have thrown in an **arbitrary** factor of 2 for aesthetics later.

3.3 Part d)

Verify that the solutions from Parts c) and a) are consistent.

The simplest ways to do this is to rewrite to the velocity profile as dimensionless units.

$$\begin{aligned}
v_z &= \left(\frac{\rho g}{2\eta}\right)\left(2\delta x - x^2]\right) \\
\frac{v_z}{\delta^{1/2}g^{1/2}} &= \frac{1}{\delta^{1/2}g^{1/2}}\left(\frac{\rho g}{2\eta}\right)\left(2\delta x - x^2\right) \\
\Pi_2 &= \frac{1}{\delta^{1/2}g^{1/2}}\left(\frac{\rho g}{2\eta}\right)\delta^2\left(2\frac{x}{\delta} - \frac{x^2}{\delta^2}\right) \\
&= \frac{1}{\delta^{1/2}g^{1/2}}\left(\frac{\rho g}{2\eta}\right)\delta^2\left(2\Pi_1 - \Pi_1^2\right) \\
&= \left(\frac{\rho\delta^{3/2}g^{1/2}}{2\eta}\right)\left(2\Pi_1 - \Pi_1^2\right)
\end{aligned}$$

$$\boxed{\Pi_2 = \frac{\Pi_1\left(2 - \Pi_1\right)}{\Pi_3}} \tag{21}$$

or in terms of the symbols that emphasize the meaning of the dimensionless variables

$$\tilde{v} = \frac{\tilde{x}\left(2 - \tilde{x}\right)}{\tilde{\eta}}. \tag{22}$$

4 Problem 4)

4.1 Part a)

Numerically determine the flow profile in a square duct.

No length was defined for the channel, just a pressure drop. The Reynolds number depends on the **pressure gradient**, therefore the pipe must be long enough that Re is not too great. Specify your length. Estimating your Re would have gotten you bonus marks.

4.2 Part b)

Compare to the series solution for flow in a rectangular duct of height h and width w, which is

$$v_x\left(y, z\right) = \frac{4h^2\Delta p}{\pi^3\eta L}\sum_{n,\text{odd}}^{\infty}\frac{1}{n^3}\left[1 - \frac{\cosh\left(\frac{n\pi y}{h}\right)}{\cosh\left(\frac{n\pi w}{2h}\right)}\right]\sin\left(\frac{n\pi z}{h}\right). \tag{23}$$

The unspecified thing here is that the range of y and z is a little odd:

$$-\frac{h}{2} \le y \le \frac{h}{2} \tag{24}$$

$$0 \le z \le w. \tag{25}$$

Another way that many students found to do a convenient shift of the coordinates was to replace $\sin\left(n\pi z/h\right)$ with $\cos\left(n\pi z/h\right)$ and use the range

$$0 \le y \le h \tag{26}$$

$$0 \le z \le w. \tag{27}$$

Using only a single term Eq. (23) produces Fig. 1

I have included here the python code that I used to to generate these graphs:

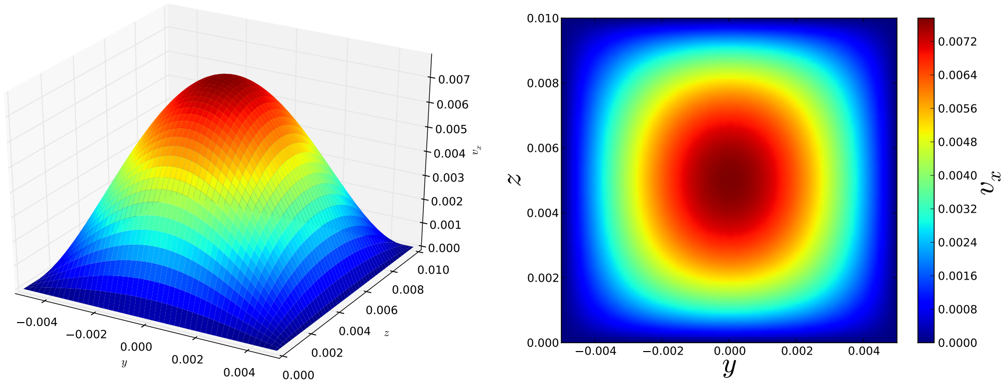

Figure 1: The low-Re velocity field for a square duct using only a single term from Eq. (23).

```
from pylab import *
from mpl_toolkits.mplot3d import Axes3D

LS = 30 # Axis number size
myMap=cm.jet # Define the color map

SZ = 100 # Number of nodes to solve on
TERMS = 1 # Number of terms to use
height = 0.01 # Channel height (meters)
width = 0.01 # Channel width (meters)
pgrad = 1.0/1.0 # Pressure gradient (Pascal/meters)
visc = 1.0E-3 # Dynamic viscosity of water

# Set the mesh
y=linspace(-0.5*height,0.5*height,SZ)
z=linspace(0.0,width,SZ)
# Initialize the velocity
vel = zeros(shape=(SZ,SZ),dtype=float)

# Compute the velocity profile
for i in range(SZ):
for j in range(SZ):
for k in range(TERMS):
n = float(2*k+1)
termk = (1.0 - cosh(n*pi*y[i]/height)/cosh(0.5*n*pi*width/height))
termk *= sin(n*pi*z[j]/height)/pow(n,3)
vel[i][j] += termk
vel[i][j] *= pgrad*(4.0*height*height)/(pi*pi*pi*visc)

# Plot as a heat map
fig1 = figure()
imshow(vel,cmap=myMap,extent=[-0.5*height,0.5*height,0.0,width],origin='lower',aspect='auto')
cb=colorbar()
cb.ax.set_ylabel(r"$v_x$", fontsize = LS)
xlabel(r"$y$", fontsize = LS)
ylabel(r"$z$", fontsize = LS)

# Plot as a 3D surface (with redundant heat map coloring)
fig2 = figure()
```

```
ax = Axes3D( fig2 )
# Turn y and z into arrays
y,z = meshgrid(y,z)
ax.plot_surface(y,z,vel,rstride=2,cstride=2,cmap=myMap,linewidth=0)
ax.set_xlabel(r"$y$")
ax.set_ylabel(r"$z$")
ax.set_zlabel(r"$v_x$")

show()
```

4.3 Part c)

Create an entrance.

An entrance can be made in two or three ways:

A large reservoir with no-slip boundaries This works fine because if the size is large enough then the gradient of the velocity profile over the cross section of the opening is negligible.

A large reservoir with slip boundaries This is significantly better (the best choice in my opinion). The velocity profile must be relatively uniform outside of the duct.

A transition from slip to no-slip boundaries in a continuous duct This is fine. It's an elegant way of simulating the geometry we wanted with out actually constructng the geometry. However, our intuition can fail us at times and programs such as COMSOL have power in that they **can** actually create the geometry of interest rather than a simplified version.

4.4 Part d)

Comment on the streamlines and pressure drop.

The most important things are that the streamlines remain ever parallel to one another and the pressure drop is extremely linear within the pipe (and hardly any drop in the reservoir).

4.5 Part e)

Estimate the entrance length.

The duct size is $h = w$. The time it takes for momentum to diffuse a distance $h/2$ is

$$y = k\sqrt{\nu t}$$
$$\frac{h}{2} = k\sqrt{\nu\tau}$$

$$\boxed{\tau = \frac{h^2}{4k^2\nu}}. \tag{28}$$

In that time the fluid travels a distance

$$\boxed{\ell = U\tau = \frac{Uh^2}{4k^2\nu}}. \tag{29}$$

This is the entrance length.

4.6 Part f)

Compare the estimate for entrance length with the numerical solution.

Those students who had low-Re found the estimate to be **extremely** accurate.