Assignment 1
Continuous Matter 4335/8191(B)

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1 Problem 1)

Consider a hemispherical vessel if radius R with a orifice of area A at the bottom. How long is required to lower the level of fluid from h_1 to h_2? For this question, we'll use Bernoulli's law

$$H = \frac{v^2}{2} + \Phi + \frac{P}{\rho} = \text{const.} \tag{1}$$

and Leonardo's law

$$A_1 v_1 = A_2 v_2. \tag{2}$$

If our coordinated system is set up so that the z-axis is normal to the fluid surface (subscript T that is at some height z with a velocity v_T) and the hole (subscript B) is at $z = 0$ then Bernoulli's law for the surface and the oriface becomes

$$\begin{aligned} H_T &= H_B \\ \frac{v_T^2}{2} + gz + \frac{P_{\text{atm}}}{\rho} &= \frac{v_B^2}{2} + g \cdot 0 + \frac{P_{\text{atm}}}{\rho} \\ \frac{v_T^2}{2} + gz &= \frac{v_B^2}{2} \\ v_B^2 &= v_T^2 + 2gz \end{aligned} \tag{3}$$

where the pressure was the atmospheric pressure P_{atm} at both the top surface and the hole. Now we solve for v_T after applying Leonardo's law

$$\begin{aligned} \left(\frac{v_T A_T}{A_B} \right)^2 &= v_T^2 + 2gz \\ v_T^2 \left[\left(\frac{A_T}{A_B} \right)^2 - 1 \right] &= 2gz \\ v_T &= \left[\frac{2gz}{\left(\frac{A_T}{A_B} \right)^2 - 1} \right]^{1/2} \end{aligned} \tag{4}$$

Now we need to recognize a few things:

1. $A_T = \pi r^2 (z)$ where we can determine the radius of the surface r as a function of z in a spherical vessel of radius R to be

$$\begin{aligned} R^2 &= r^2 + (R - z)^2 \\ r^2 &= R^2 - \left(R^2 - 2Rz + z^2 \right) \\ &= 2Rz - z^2 \end{aligned} \tag{5}$$

which means that the velocity of the surface written as a function of surface height is

$$v_T = \left[\frac{2gz}{\left(\frac{\pi(2Rz-z^2)}{A_B}\right)^2 - 1} \right]^{1/2} \tag{6}$$

2. The velocity if the surface is simply $v_T = \partial z/\partial t$ so then to find the position of the surface we integrate

$$\begin{aligned} \int_0^{\Delta t} dt &= \int_{h_1}^{h_2} \frac{1}{v_T} dz \\ &= \int_{h_1}^{h_2} \left[\frac{\left(\frac{\pi(2Rz-z^2)}{A_B}\right)^2 - 1}{2gz} \right]^{1/2} dz \end{aligned} \tag{7}$$

This integral is not very fun.

3. The integral becomes tenable if we make the fairly reasonable assumption that $A_T/A_B \gg 1$. Of course this can't be true for all heights h_i but it if the oriface is small (not said in the question) then this is a good assumption. Then the velocity within the integral can be made simplier

$$\begin{aligned} \int_0^{\Delta t} dt &= \int_{h_1}^{h_2} \left[\frac{\left(\frac{\pi(2Rz-z^2)}{A_B}\right)^2 - 1}{2gz} \right]^{1/2} dz \\ \Delta t &\approx \int_{h_1}^{h_2} \left[\frac{\left(\frac{\pi(2Rz-z^2)}{A_B}\right)^2}{2gz} \right]^{1/2} dz \\ &= \int_{h_1}^{h_2} \left(\frac{1}{2g}\right)^{1/2} \left(\frac{\pi}{A_B}\right) \frac{(2Rz - z^2)}{z^{1/2}} dz \\ &= \left(\frac{\pi}{A_B\sqrt{2g}}\right) \int_{h_1}^{h_2} \left(2Rz^{1/2} - z^{3/2}\right) dz \\ &= \left(\frac{\pi}{A_B\sqrt{2g}}\right) \left[\frac{4}{3}Rz^{3/2} - \frac{2}{5}z^{5/2}\right]_{h_1}^{h_2} \end{aligned}$$

$$\boxed{\Delta t = \left(\frac{2\pi}{A_B\sqrt{2g}}\right) \left[\frac{2}{3}R\left(h_1^{3/2} - h_2^{3/2}\right) - \frac{1}{5}\left(h_1^{5/2} - h_2^{5/2}\right)\right]}. \tag{8}$$

2 Problem 2)

If you look at the schematic on the Problem Set you will see an irregularly shaped tank filled to a height h_1. There is a weak spot at $*$ that will break if $p_* > p_{\max}$. The thin part has a radius r_1 and is open to atmosphere. There is some $h_1^{\max}$ which corresponds to $p_*^{\max}$ at point $*$. Notice that I've used JUST SLIGHTLY different notation than the question – which I hope will be a touch clearer.

2.1 Part a)

Derive an expression for $h_1^{\max}$.

Once again we use Bernoulli's theorem for incompressible fluids (Eq. (1)):

$$H = \frac{v^2}{2} + \Phi + \frac{p}{\rho}.$$

At all times $v = 0$ in this part of the question. At the thin part of the tank we have

$$H_1 = gh_1 + \frac{p_{\text{atm}}}{\rho}, \tag{9}$$

while at the weak part we have

$$H_* = gh_* + \frac{p_*}{\rho} \tag{10}$$

They are equal at all times and when $p_* = p_*^{\max}$ then $h_1 = h_1^{\max}$:

$$gh_1^{\max} + \frac{p_{\text{atm}}}{\rho} = gh_* + \frac{p_*^{\max}}{\rho}$$

$$\boxed{h_1^{\max} = h_* + \left(\frac{p_*^{\max} - p_{\text{atm}}}{\rho g}\right)}. \tag{11}$$

2.2 Part b)

A hole of area A_* forms at $*$. What velocity will the fluid exit from the hole at immediately after the break occurs?

Now suddenly the velocity does not have to be $v = 0$ so we reformulate Bernoulli's equation at each point

$$H_1 = H_*$$
$$\frac{v_1^2}{2} + gh_1^{\max} + \frac{p_{\text{atm}}}{\rho} = \frac{v_*^2}{2} + gh_*^{\max} + \frac{p_*^{\max}}{\rho}$$

A couple of things to notice

1. As soon as the break occurs $p_*^{\max} \to p_{\text{atm}}$
2. We can use $h_1^{\max}$ from part a)
3. The part of the tank where h_1 is was described as "thin" and of radius r_1. Therefore, we shouldn't expect that v_1 is negligible unless the hole A_* is assumed to be much smaller but there is no need to do this since we can once again use Leonardo's law (Eq. (2)).

Keeping these points in mind we continue.

$$\begin{aligned}
H_1 &= H_* \\
\frac{v_1^2}{2} + gh_1^{\max} + \frac{p_{\text{atm}}}{\rho} &= \frac{v_*^2}{2} + gh_*^{\max} + \frac{p_*^{\max}}{\rho} \\
\frac{v_1^2}{2} + g\left[h_* + \left(\frac{p_*^{\max} - p_{\text{atm}}}{\rho g}\right)\right] + \frac{p_{\text{atm}}}{\rho} &= \frac{v_*^2}{2} + gh_*^{\max} + \frac{p_{\text{atm}}}{\rho} \\
\frac{v_1^2}{2} + \frac{p_*^{\max}}{\rho} &= \frac{v_*^2}{2} + \frac{p_{\text{atm}}}{\rho} \\
\frac{v_*^2}{2} - \frac{v_1^2}{2} &= \frac{p_*^{\max}}{\rho} - \frac{p_{\text{atm}}}{\rho}
\end{aligned}$$

$$\frac{v_*^2}{2} - \left(\frac{A_*}{\pi r_1^2}\right)\frac{v_*^2}{2} = \frac{p_*^{\max}}{\rho} - \frac{p_{\text{atm}}}{\rho}$$

$$v_*^2 = \frac{2}{\rho}\left(\frac{p_*^{\max} - p_{\text{atm}}}{1 - \left(\frac{A_*}{\pi r_1^2}\right)^2}\right)$$

$$\boxed{v_* = \left[\frac{2}{\rho}\left(\frac{p_*^{\max} - p_{\text{atm}}}{1 - \left(\frac{A_*}{\pi r_1^2}\right)^2}\right)\right]^{1/2}}. \tag{12}$$

3 Problem 3)

Consider the following velocity field

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} k_1 x e^{-k_3 t} \\ k_2 y \\ 0 \end{pmatrix}. \tag{13}$$

3.1 Part a)

Find the equation that represents the streamlines. You can let $k_1 = k_2 = k_3 = 1$ to simplify things.

The family of streamlines at time $t > 0$ are the solutions of

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}.$$

Since v_z doesn't vary, it does not concern us.

$$\frac{dy'}{v_y} = \frac{dx'}{v_x}$$

$$\frac{dy'}{k_2 y'} = \frac{dx'}{k_1 x' e^{-k_3 t}}$$

$$\int_{y_0}^{y} \frac{dy'}{k_2 y'} = \int_{x_0}^{x} \frac{dx'}{k_1 x' e^{-k_3 t}}$$

$$\int_{y_0}^{y} \frac{dy'}{y'} = \frac{k_2 e^{k_3 t_0}}{k_1} \int_{x_0}^{x} \frac{dx'}{x'}$$

$$\ln\left(\frac{y}{y_0}\right) = \frac{k_2 e^{k_3 t_0}}{k_1} \ln\left(\frac{x}{x_0}\right)$$

$$\boxed{\ln\left(\frac{y}{y_0}\right) = e^{t_0} \ln\left(\frac{x}{x_0}\right)}. \tag{14}$$

3.2 Part b)

What is the pathline for a fluid particle coincident with the point (x_0, y_0) at $t = 0$?

Integrating the velocities gives the trajectories.

$$v_x = \frac{\partial x}{\partial t} = k_1 x' e^{-k_3 t}$$

$$\int \frac{dx}{x} = \int k_1 e^{-k_3 t} dt$$

$$\ln x = c - \frac{k_1}{k_3} e^{-k_3 t}$$

At $t = t_0$ (we'll say $t_0 = 0$ in second), $x = x_0$ so

$$c = \ln x_0 + \frac{k_1}{k_3} e^{-k_3 t_0},$$

which immediately leads to

$$\ln\left(\frac{x}{x_0}\right) = \frac{k_1}{k_3}\left(e^{-k_3 t_0} - e^{-k_3 t}\right)$$

$$\boxed{x = x_0 \exp\left[\frac{k_1}{k_3}\left(e^{-k_3 t_0} - e^{-k_3 t}\right)\right]}. \tag{15}$$

We can do exactly the same thing for the y-component (but a little quicker now)

$$\frac{\partial y}{\partial t} = k_2 y$$
$$\ln y = k_2 t + c$$
$$\ln\left(\frac{y}{y_0}\right) = k_2 (t - t_0)$$

$$\boxed{y = y_0 \exp\left[k_2 (t - t_0)\right]}. \tag{16}$$

Letting $k_1 = k_2 = k_3 = 1$ and $t_0 = 0$ we get

$$\boxed{\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \exp\left[1 - e^{-t}\right] \\ y_0 e^t \end{pmatrix}}. \tag{17}$$

3.3 Part c)

Find the equation of the streakline passing through a given point $\vec{r} = a\hat{i} + b\hat{j}$.

We know that streaklines will change in time so we say that the streakline goes through $\vec{r}$ at some time τ which we will have to get rid of later. Our job is to relate x and y to find the curve that represents the streakline through $\vec{r}$. Then using the conclusions of Part b) we can say the trajectory to get to that point is

$$x = a \exp\left[\frac{k_1}{k_3}\left(e^{-k_3 \tau} - e^{-k_3 t}\right)\right] \tag{18}$$

$$y = b \exp\left[k_2 (t - \tau)\right]. \tag{19}$$

All we did was let $x_0 \to a$, $y_0 \to b$ and $t_0 \to \tau$. But we don't know **when** the fluid goes through $\vec{r}$ so we eliminate τ using Eq. (19)

$$\frac{y}{b} = e^{k_2 t} e^{-k_2 \tau}$$
$$e^{-\tau} = \left(\frac{y}{b}\right)^{1/k_2} e^{-t},$$

which we can substitute into the equation for x thus relating x and y as

$$x = a \exp\left[\frac{k_1}{k_3}\left(e^{-k_3\tau} - e^{-k_3 t}\right)\right]$$
$$x = a \exp\left[\frac{k_1}{k_3}\left(\left(e^{-\tau}\right)^{k_3} - e^{-k_3 t}\right)\right]$$
$$x = a \exp\left[\frac{k_1}{k_3}\left(\left(\left(\frac{y}{b}\right)^{1/k_2} e^{-t}\right)^{k_3} - e^{-k_3 t}\right)\right]$$
$$x = a \exp\left[\frac{k_1}{k_3}\left(\left(\frac{y}{b}\right)^{k_3/k_2} e^{-k_3 t} - e^{-k_3 t}\right)\right]$$
$$x = a \exp\left[\frac{k_1}{k_3} e^{-k_3 t}\left(\left(\frac{y}{b}\right)^{k_3/k_2} - 1\right)\right]$$

$$\boxed{\ln\left(\frac{x}{a}\right) = \frac{k_1}{k_3} e^{-k_3 t}\left[\left(\frac{y}{b}\right)^{k_3/k_2} - 1\right]}. \tag{20}$$

4 Problem 4)

Consider this velocity field

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 16x^2 + y \\ 10 \\ yz^2 \end{pmatrix}. \tag{21}$$

Determine the circulation around the contour

$$\begin{cases} 0 \leq x \leq 10 & \text{at } y = 0 \\ 0 \leq y \leq 5 & \text{at } x = 10 \\ 0 \leq x \leq 10 & \text{at } y = 5 \\ 0 \leq y \leq 5 & \text{at } x = 0. \end{cases}$$

4.1 Choice 1

The straightforward way:

$$\Gamma = \oint_c \vec{v} \cdot d\ell \tag{22}$$
$$= \int_0^{10} v_x dx\bigg|_{y=0} + \int_0^5 v_y dy\bigg|_{x=10} + \int_{10}^0 v_x dx\bigg|_{y=5} + \int_5^0 v_y dy\bigg|_{x=0}$$
$$= \int_0^{10} \left(16x^2 + 0\right) dx + \underbrace{\int_0^5 10dy} + \int_{10}^0 \left(16x^2 + 5\right) dx + \underbrace{\int_5^0 10dy}$$
$$= \int_0^{10} \left(16x^2\right) dx + \int_{10}^0 \left(16x^2 + 5\right) dx$$
$$= \underbrace{\int_0^{10} \left(16x^2\right) dx} + \underbrace{\int_{10}^0 \left(16x^2\right) dx} + \int_{10}^0 5dx$$
$$= \int_{10}^0 5dx$$
$$= -50. \tag{23}$$

4.2 Choice 2

The vorticity way:

The area is

$$\vec{S} = 10 \times 5 \times \hat{k} = 50\hat{k}. \tag{24}$$

Therefore, since the circulations is

$$\Gamma = \int_S \vec{\omega} \cdot d\vec{S} \tag{25}$$

we only need the $\hat{k}$-component of the vorticity, which is

$$\begin{aligned}\omega_z &= \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \\ &= \frac{\partial}{\partial x}(10) - \frac{\partial null}{\partial y}\left(16x^2 + y\right) \\ &= 0 - 1 = -1,\end{aligned} \tag{26}$$

which makes the circulation simply

$$\boxed{\Gamma = \int_S \vec{\omega} \cdot d\vec{S} = (-1)(50) = -50}. \tag{27}$$

5 Problem 5

Consider the following velocity field

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \frac{-2xyz}{(x^2+y^2)^2} \\ \frac{(x^2+y^2)z}{(x^2+y^2)^2} \\ \frac{y}{x^2+y^2} \end{pmatrix} \tag{28}$$

5.1 Part a)

Is the flow incompressible?

We can tell from the continuity equation

$$\begin{aligned}\frac{D\rho}{Dt} &= 0 \\ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) &= 0.\end{aligned}$$

Assuming that the density is constant this becomes simply

$$\begin{aligned}\vec{\nabla} \cdot \vec{v} &= 0 \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} &= 0\end{aligned} \tag{29}$$

Let's do each derivative separately:

- $$\frac{\partial v_z}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{x^2 + y^2}\right) = 0$$

- $$\frac{\partial v_y}{\partial y} = \frac{\partial}{\partial z}\left(\frac{\left(x^2 + y^2\right) z}{\left(x^2 + y^2\right)^2}\right) = 2yz\left[\frac{y^2 - 3x^2}{\left(x^2 + y^2\right)^3}\right]$$

- $$\frac{\partial v_x}{\partial x} = \frac{\partial}{\partial z}\left(\frac{-2xyz}{(x^2+y^2)^2}\right) = -2yz\left[\frac{y^2-3x^2}{(x^2+y^2)^3}\right]$$

Therefore

$$\boxed{\vec{\nabla}\cdot\vec{v} = -2yz\left[\frac{y^2-3x^2}{(x^2+y^2)^3}\right] + 2yz\left[\frac{y^2-3x^2}{(x^2+y^2)^3}\right] + 0 = 0} \tag{30}$$

and we conclude that the fluid is incompressible.

5.2 Part b)

What is the vorticity? The vorticity is the curl of the velocity

$$\begin{aligned}\vec{\omega} &= \vec{\nabla}\times\vec{v}\\ &= \begin{pmatrix}\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\end{pmatrix}\end{aligned}$$

If you do all these derivatives (I'm not going to bother typing them out) you will find that each component is zero *i.e.*

$$\boxed{\vec{\omega} = \vec{0}}. \tag{31}$$

6 Problem 6)

Consider an infinite plate that separates two fluids. Each fluid exerts a force on the plate due to surface tension. Correspondingly, the height of the fluid interface is different on each side. We take as known that the heights are given by

$$h_1(x) = L_c \cot\theta_1 e^{x/L_c} \tag{32}$$

$$h_2(x) = L_c\left(\frac{\cot\theta_1\cosh\left(\frac{d-x}{L_c}\right) + \cot\theta_2\cosh\left(\frac{x}{L_c}\right)}{\sinh\left(\frac{d}{L_c}\right)}\right) \tag{33}$$

where the angles are fixed and x is the distance from the plate.

6.1 Part a)

Determine the horizontal force per unit length on the plate.

The surface tension term $\gamma\times$ the curvature is the action of the interface on the bulk fluid. It changes the pressure such that fluid can rise to some height against the gravitational force down which leads to

$$\gamma\frac{d^2h}{dx^2} = \rho g h, \tag{34}$$

But the change in pressure isn't directional (*i.e.* it doesn't just push the fluid up). It's isotropic (*i.e.* it pushes on the container/plate as well). So the surface tension force acting on each side of the plate is

$$F_i = \int_{-\infty}^{h_i}\gamma_i\frac{\partial^2 z}{\partial x^2}dz, \tag{35}$$

which means that the net force acting on the plate is

$$
\begin{aligned}
F &= F_1 + F_2 \\
&= -\int_{-\infty}^{h_2} \gamma \frac{\partial^2 z}{\partial x^2} dz + \int_{-\infty}^{h_1} \gamma \frac{\partial^2 z}{\partial x^2} dz \\
&= -\int_{-\infty}^{h_2} \gamma \frac{\partial^2 z}{\partial x^2} dz - \int_{h_1}^{-\infty} \gamma \frac{\partial^2 z}{\partial x^2} dz \\
&= -\int_{-\infty}^{h_2} \rho g z dz - \int_{h_1}^{-\infty} \rho g z dz \\
&= -\int_{h_1}^{h_2} \rho g z dz
\end{aligned}
$$

$$
\boxed{F = \left(\frac{\rho g}{2}\right) \left[h_1^2 - h_2^2\right]}. \tag{36}
$$

Since we are interested in the force **at the wall** $h_i(x)$ must be evaluated at $x = 0$ which leads to (also recalling that the question stated $L_c = \sqrt{\gamma/\rho g}$)

$$
\begin{aligned}
F &= \left(\frac{\rho g}{2}\right) \left[h_1^2(0) - h_2^2(0)\right] \\
&= \left(\frac{\rho g}{2}\right) \left[L_c^2 \cot^2\theta_1 e^0 - L_c^2 \left(\frac{\cot\theta_1 \cosh\left(\frac{d-0}{L_c}\right) + \cot\theta_2 \cosh\left(\frac{0}{L_c}\right)}{\sinh\left(\frac{d}{L_c}\right)}\right)^2\right] \\
&= \left(\frac{\rho g}{2}\right) \left[L_c^2 \cot^2\theta_1 - L_c^2 \frac{\left(\cot\theta_1 \cosh\left(\frac{d}{L_c}\right) + \cot\theta_2\right)^2}{\sinh^2\left(\frac{d}{L_c}\right)}\right] \\
&= \left(\frac{\rho g}{2}\right) \left(\frac{\gamma}{\rho g}\right) \left[\cot^2\theta_1 - \frac{\left(\cot\theta_1 \cosh\left(\frac{d}{L_c}\right) + \cot\theta_2\right)^2}{\sinh^2\left(\frac{d}{L_c}\right)}\right]
\end{aligned}
$$

$$
\boxed{\frac{F}{\gamma} = \frac{1}{2} \left[\cot^2\theta_1 - \frac{\left(\cot\theta_1 \cosh\left(\frac{d}{L_c}\right) + \cot\theta_2\right)^2}{\sinh^2\left(\frac{d}{L_c}\right)}\right]} \tag{37}
$$