Assignment 8

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1 Goldstein Chapter 9

Canonical Transformations

1.1 Problem 9.15

Consider the consider the transformation from coordinates x, p to generalized coordinates Q, P

$$Q = \frac{\alpha p}{x} \tag{1a}$$
$$P = \beta x^2 \tag{1b}$$

where α, β are constants. Determine when this represents a canonical transformation, find a generating function and apply the transformation to the solution of the simple harmonic oscillator.

1.1.1 Canonical?

We can reverse the transformations in three ways

$$x = \frac{\alpha p}{Q} \tag{2a}$$
$$p = \frac{xQ}{\alpha} \tag{2b}$$
$$x = \sqrt{\frac{P}{\beta}} \tag{2c}$$

We know that a transformation is canonical only if

$$\left.\frac{\partial Q_i}{\partial q_j}\right|_{q,p} = \left.\frac{\partial p_j}{\partial P_i}\right|_{Q,P}$$
$$\left.\frac{\partial Q_i}{\partial p_j}\right|_{q,p} = -\left.\frac{\partial q_j}{\partial P_i}\right|_{Q,P}$$
$$\left.\frac{\partial P_i}{\partial q_j}\right|_{q,p} = -\left.\frac{\partial p_j}{\partial Q_i}\right|_{Q,P}$$
$$\left.\frac{\partial P_i}{\partial p_j}\right|_{q,p} = \left.\frac{\partial q_j}{\partial Q_i}\right|_{Q,P}.$$

The derivatives of interest are

$$\frac{\partial Q}{\partial x} = -\frac{\alpha p}{x^2} \qquad \frac{\partial p}{\partial P} = \frac{\partial}{\partial P}\frac{xQ}{\alpha} = \frac{Q}{\alpha}\frac{\partial}{\partial P}\sqrt{\frac{P}{\beta}} = \frac{1}{2}\frac{Q}{\alpha}\frac{1}{\sqrt{\beta P}} = \frac{1}{2}\frac{Q}{\alpha}\frac{1}{\beta x}$$

$$\frac{\partial Q}{\partial p} = \frac{\alpha}{x} \qquad \frac{\partial x}{\partial P} = \frac{\partial}{\partial P}\sqrt{\frac{P}{\beta}} = \frac{1}{2}\frac{1}{\sqrt{\beta P}} = \frac{1}{2\beta x}$$

$$\frac{\partial P}{\partial x} = 2\beta x \qquad \frac{\partial p}{\partial Q} = \frac{x}{\alpha}$$

$$\frac{\partial x}{\partial Q} = -\frac{\alpha p}{Q^2} \qquad \frac{\partial P}{\partial p} = \beta\frac{\partial x^2}{\partial p} = \beta\frac{\partial}{\partial p}\left(\frac{\alpha p}{Q}\right)^2 = 2\frac{\beta\alpha^2}{Q^2}p = 2\frac{\beta\alpha^2}{Q^2}\frac{xQ}{\alpha} = 2\frac{\alpha\beta x}{Q}$$

Therefore, the equations become

$$-\frac{\alpha p}{x^2} = \frac{1}{2}\frac{Q}{\alpha}\frac{1}{\beta x}$$

$$\frac{\alpha}{x} = -\frac{1}{2\beta x}$$

$$2\beta x = -\frac{x}{\alpha}$$

$$2\frac{\alpha\beta x}{Q} = -\frac{\alpha p}{Q^2} = -\frac{x^2}{\alpha p}.$$

The two middle equations clearly demand that

$$\boxed{2\alpha\beta = -1} \tag{3}$$

and it turns out that the first and last equation (which are also the same) can become

$$-\frac{\alpha p}{x^2} = \frac{1}{2}\frac{Q}{\alpha}\frac{1}{\beta x} = \frac{1}{2}\frac{\alpha p}{\alpha x}\frac{1}{\beta x} = \frac{p}{2\beta x^2}$$

$$-2\alpha\beta = \frac{px^2}{x^2 p}$$

the exact same condition.

1.1.2 Generating Function

The generating function is some function $F(x, Q, t)$ for which

$$p_i = \frac{\partial F}{\partial q_i} \qquad P_i = -\frac{\partial F}{\partial Q_i}$$

$$Q_i = \frac{\partial F}{\partial P_i} \qquad q_i = -\frac{\partial F}{\partial p_i}.$$

Focusing on momenta p, P and recalling our two transformations in Eq. (1) gives

$$p = \frac{xQ}{\alpha} = \frac{\partial F}{\partial x}$$

$$F(x, Q) - F(x, 0) = \int_0^x \frac{xQ}{\alpha}dx = \frac{Qx^2}{2\alpha} + f_0(Q)$$

where we have been quite general and left the possiblity for some function of Q only. The transformed momentum gives

$$\begin{aligned} P = \beta x^2 &= -\frac{\partial F}{\partial Q} \\ F(x,Q) - F(0,Q) &= -\int_0^Q \beta x^2 dQ = -\beta x^2 Q + f_1(x) \\ &= \frac{x^2 Q}{2\alpha} + f_1(x) \end{aligned}$$

Comparing the two generating functions it is clear that neither f_0 nor f_1 can be allowed (unless it's some constant). So we conclude

$$\boxed{F(x,Q) = \frac{x^2 Q}{2\alpha} + c_0} \tag{4}$$

1.1.3 Harmonic Oscillator

The solution for a harmonic oscillator is

$$x(t) = \sqrt{\frac{2E}{m\omega^2}} \cos(\omega t + \phi) \tag{5}$$

$$p(t) = m\dot{x} = -\sqrt{2mE} \sin(\omega t + \phi) \tag{6}$$

and all the question asks us to do is transform the solution according to Eq. (1) . So

$$\begin{aligned} Q &= \frac{\alpha p}{x} \\ &= -\frac{\alpha\sqrt{2mE}\cos(\omega t + \phi)}{\sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \phi)} \end{aligned}$$

$$\boxed{Q = -\alpha m\omega \tan(\omega t + \phi)} \tag{7}$$

and

$$P = \beta x^2$$

$$\boxed{P = \beta \frac{2E}{m\omega^2} \cos^2(\omega t + \phi)}. \tag{8}$$