Assignment 7

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1 Marion and Thornton Chapter 12

Coupled Oscillations

1.1 Problem 12.16

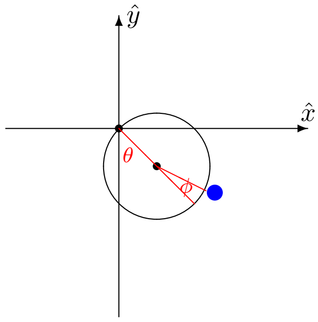

Figure 1: Hoop hanging from a fixed point with a particle confined to move along it's edge.

As shown in Fig. 1 consider a hoop of mass M that can swing on a fixed pivot point. Let there be a point particle free to move along the hoop in a frictionless manner. What are the eigenfrequencies, assuming only small oscillations?

Our purpose is to write the energies as

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k \tag{1}$$

$$U = \frac{1}{2}\sum_{j,k} A_{jk}q_j q_k. \tag{2}$$

After we have done so the equations of motion are simply

$$\sum_j \left(A_{jk} - \omega^2 m_{jk}\right) a_j = 0 \tag{3}$$

which must satisfy

$$\left|\tilde{A} - \omega^2\tilde{m}\right| = 0. \tag{4}$$

Then for our specific case, let's realize that the centre of the hoop's coordinates are

$$\begin{aligned}
x_h(\theta) &= R\sin\theta \\
y_h(\theta) &= -R\cos\theta \\
\dot{x}_h(\theta) &= R\dot{\theta}\cos\theta \\
\dot{y}_h(\theta) &= R\dot{\theta}\sin\theta
\end{aligned}$$

relative to the pivot point and the particle's coordinates are

$$\begin{aligned}
x_p(\theta,\phi) &= x_h + R\sin(\theta+\phi) \\
&= R\left[\sin\theta + \sin(\theta+\phi)\right] \\
y_p(\theta,\phi) &= y_h - R\cos(\theta+\phi) \\
&= -R\left[\cos\theta + \cos(\theta+\phi)\right] \\
\dot{x}_p(\theta,\phi) &= \dot{x}_h + R\left(\dot{\theta}+\dot{\phi}\right)\cos(\theta+\phi) \\
&= R\left(\dot{\theta}\cos\theta + \left(\dot{\theta}+\dot{\phi}\right)\cos(\theta+\phi)\right) \\
\dot{y}_h(\theta,\phi) &= \dot{y}_h + R\left(\dot{\theta}+\dot{\phi}\right)\sin(\theta+\phi) \\
&= R\left(\dot{\theta}\sin\theta + \left(\dot{\theta}+\dot{\phi}\right)\sin(\theta+\phi)\right)
\end{aligned}$$

Then the potential energy of the hoop and the mass (assuming $\cos\theta \approx 1-\theta^2/2+\ldots$) is

$$\begin{aligned}
U' &= Mgy_h + Mgy_p = -MgR\left[2\cos\theta + \cos(\theta+\phi)\right] \\
&= MgR\left[2\left(1-\frac{\theta^2}{2}+\ldots\right)+\left(1-\frac{(\theta+\phi)^2}{2}+\ldots\right)\right] \\
&\approx MgR\left[3-\theta^2-\frac{\theta^2+2\theta\phi+\phi^2}{2}\right] \\
&= \frac{MgR}{2}\left[6-3\theta^2-2\theta\phi-\phi^2\right] \\
U &= -\frac{MgR}{2}\left[3\theta^2+2\theta\phi+\phi^2\right]
\end{aligned}$$

where we dumped the $3mgR$ because constant values in potentials are meaningless. If we define $\vec{q} = [\theta, \phi]$ then we can rewrite the potential energy in the beautiful form of Eq. (2)

$$U = \frac{1}{2}\vec{q}^T\tilde{A}\vec{q} \tag{5}$$

where $\tilde{A}$ is the important tensor we were searching for:

$$\boxed{\tilde{A} = MgR\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}}. \tag{6}$$

Now on to the kinetic energy. Again expand trigonometric functions to first order to get

$$
\begin{aligned}
T = T_h + T_p &= \frac{I_{\rm cm} + I_{\rm off}}{2}\dot{\theta}^2 + \frac{M}{2}\left(\dot{x}_p^2 + \dot{y}_p^2\right) \\
&= \frac{MR^2}{2}\dot{\theta}^2 + \frac{MR^2}{2}\dot{\theta}^2 + \frac{MR^2}{2}\left[\left(\dot{\theta}\cos\theta + \left(\dot{\theta}+\dot{\phi}\right)\cos\left(\theta+\phi\right)\right)^2 + \left(\dot{\phi}\sin\phi + \left(\dot{\theta}+\dot{\phi}\right)\sin\left(\theta+\phi\right)\right)^2\right] \\
&\approx MR^2\dot{\theta}^2 + \frac{MR^2}{2}\left[\left(\dot{\theta}\left(1-\frac{\theta^2}{2}\right) + \left(\dot{\theta}+\dot{\phi}\right)\left(1-\frac{\theta^2+2\theta\phi+\phi^2}{2}\right)\right)^2\right. \\
&\quad \left. + \left(\dot{\phi}\left(\phi-\frac{\phi^3}{3}\right) + \left(\dot{\theta}+\dot{\phi}\right)\left(\theta+\phi-\frac{\left(\theta+\phi\right)^3}{3}\right)\right)^2\right].
\end{aligned}
$$

Since oscillations are small we only keep terms to second order in the velocities (as we did for position) which leads to

$$
\begin{aligned}
T &\approx MR^2\dot{\theta}^2 + \frac{MR^2}{2}\left[\left(\dot{\theta} + \left(\dot{\theta}+\dot{\phi}\right)\right)^2 + \left(\dot{\phi}\phi + \left(\dot{\theta}+\dot{\phi}\right)\left(\theta+\phi\right)\right)^2\right] \\
&= MR^2\left\{\dot{\theta}^2 + \frac{1}{2}\left[\left(2\dot{\theta}+\dot{\phi}\right)^2 + \left(\dot{\phi}\phi + \dot{\theta}\theta + \dot{\theta}\phi + \dot{\phi}\theta + \dot{\phi}\phi\right)^2\right]\right\} \\
&= MR^2\left\{\dot{\theta}^2 + \frac{1}{2}\left[4\dot{\theta}^2 + 4\dot{\theta}\dot{\phi} + \dot{\phi}^2 + \underbrace{\left(\dot{\phi}\phi + \dot{\theta}\theta + \dot{\theta}\phi + \dot{\phi}\theta + \dot{\phi}\phi\right)^2}_{\approx 0}\right]\right\} \\
&\approx \frac{MR^2}{2}\left\{2\dot{\theta}^2 + 4\dot{\theta}^2 + 4\dot{\theta}\dot{\phi} + \dot{\phi}^2\right\} \\
&= \frac{MR^2}{2}\left\{6\dot{\theta}^2 + 4\dot{\theta}\dot{\phi} + \dot{\phi}^2\right\}
\end{aligned}
$$

which, just like U, we also want to rewrite in a tensor form like Eq. (1) as $\vec{q} = [\theta, \phi]$ and $\dot{\vec{q}} = \left[\dot{\theta}, \dot{\phi}\right]$ in

$$T = \frac{1}{2}\dot{\vec{q}}^T\tilde{m}\dot{\vec{q}} \tag{7}$$

where

$$\boxed{\tilde{m} = MR^2\begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix}}. \tag{8}$$

We are now poised to find the eigenvalues through Eq. (4)

$$
\begin{aligned}
0 &= \left|\tilde{A} - \omega^2\tilde{m}\right| \\
&= \begin{vmatrix} 3MgR - 6MR^2\omega^2 & MgR - 2MR^2\omega^2 \\ MgR - 2MR^2\omega^2 & MgR - MR^2\omega^2 \end{vmatrix} \\
&= MR\begin{vmatrix} 3g - 6R\omega^2 & g - 2R\omega^2 \\ g - 2R\omega^2 & g - R\omega^2 \end{vmatrix} \\
&= MR\left[\left(3g - 6R\omega^2\right)\left(g - R\omega^2\right) - \left(g - 2R\omega^2\right)\left(g - 2R\omega^2\right)\right] \\
&= MR\left[3g^2 - 3gR\omega^2 - 6gR\omega^2 + 6R^2\omega^4 - g^2 + 4gR\omega^2 - 4R^2\omega^4\right] \\
&= MR\left[2g^2 - 5gR\omega^2 + 2R^2\omega^4\right]
\end{aligned}
$$

Therefore, we know the eigenfrequencies are

$$\begin{aligned}\omega^2 &= \frac{5gR \pm \sqrt{(5gR)^2 - 4\left(2R^2\right)\left(2g^2\right)}}{4R^2} \\ &= \frac{5gR \pm \sqrt{25g^2R^2 - 16R^2g^2}}{4R^2} \\ &= \frac{5gR \pm 3gR}{4R^2}\end{aligned}$$

$$\boxed{\omega^2 = \begin{cases} 2\frac{g}{R} \\ \frac{1}{2}\frac{g}{R} \end{cases}}. \tag{9}$$

These eigenvalues correspond to eigenvectors from Eq. (3) . In general,

$$\begin{aligned}0 &= \left[\tilde{A} - \omega^2\tilde{m}\right]\cdot\vec{a} \\ &= \left[MgR\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix} - \omega^2MR^2\begin{pmatrix}6 & 2\\ 2 & 1\end{pmatrix}\right]\cdot\vec{a} \\ &= MR\begin{pmatrix}3g - 6\omega^2R & g - 2\omega^2R\\ g - 2\omega^2R & g - \omega^2R\end{pmatrix}\cdot\vec{a}\end{aligned} \tag{10}$$

Consider first $\omega_1^2 = 2g/R$ which reduces Eq. (10) to

$$\begin{aligned}0 &= \begin{pmatrix}3g - 6\frac{2g}{R}R & g - 2\frac{2g}{R}R\\ g - 2\frac{2g}{R}R & g - \frac{2g}{R}R\end{pmatrix}\cdot\vec{a}_1 \\ &= \begin{pmatrix}3g - 12g & g - 4g\\ g - 4g & g - 2g\end{pmatrix}\cdot\vec{a}_1 \\ &= \begin{pmatrix}-9 & -3\\ -3 & -1\end{pmatrix}\cdot\vec{a}_1 \\ &= -9a_{1,1} - 3a_{1,2} - 3a_{1,1} - a_{1,2} \\ &= 12a_{1,1} + 4a_{1,2}\end{aligned}$$

$$\boxed{\vec{a_1} = \begin{pmatrix}1\\ -3\end{pmatrix}}. \tag{11}$$

Next consider $\omega_2^2 = g/2R$ for which Eq. (10) gives

$$\begin{aligned}0 &= \begin{pmatrix}3g - 6\frac{g}{2R}R & g - 2\frac{g}{2R}R\\ g - 2\frac{g}{2R}R & g - \frac{g}{2R}R\end{pmatrix}\cdot\vec{a}_2 \\ &= \begin{pmatrix}3g - 3g & g - g\\ g - g & g - \frac{g}{2}\end{pmatrix}\cdot\vec{a}_2 \\ &= \begin{pmatrix}0 & 0\\ 0 & \frac{1}{2}\end{pmatrix}\cdot\vec{a}_2 \\ &= \frac{a_{2,2}}{2}\end{aligned}$$

$$\boxed{\vec{a_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}. \tag{12}$$

The solutions $\vec{q}$ are linear combination of these roots. The coordinates and the eigenvectors are always connected through the normal coordinates $\vec{\eta}$ by

$$q_j(t) = \sum_r a_{jr}\eta_r(t) \tag{13}$$

which means for us that

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \sum_r a_{j,r}\eta_r$$
$$\theta = a_{1,1}\eta_1 + a_{2,1}\eta_2$$
$$\phi = a_{1,2}\eta_1 + a_{2,2}\eta_2.$$

In order for the oscillations to occur at the first eigenfrequency ω_1 we require the η_2 contribution to be zero *i.e.* $\eta_2 = 0$ which gives inital conditions:

$$\theta = a_{1,1}\eta_1 = \eta_1 \tag{14a}$$
$$\phi = a_{1,2}\eta_1 = -3\eta_1 = -3\theta. \tag{14b}$$

Likewise, to obtain the second eigenfrequency set $\eta_1 = 0$ to get

$$\theta = a_{2,1}\eta_2 = \eta_2 \neq 0 \tag{15a}$$
$$\phi = a_{2,2}\eta_2 = 0 \times \eta_2 = 0. \tag{15b}$$

This is an interesting situation. It corresponds to the hoop swinging and the mass staying at the same spot on the hoop! Return to Fig. 1 to see this. Cool, eh?

1.2 Problem 12.16 Other option

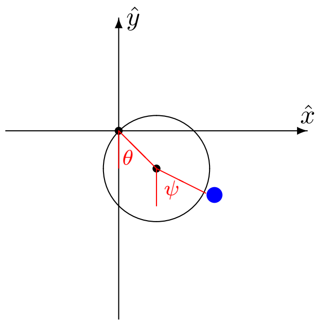

Figure 2: Hoop hanging from a fixed point with a particle confined to move along it's edge.

We repeat the other calculation for a new definition of the bottom angle ψ.

The hoop's coordinates are unchanged

$$\begin{aligned}
x_h(\theta) &= R\sin\theta \\
y_h(\theta) &= -R\cos\theta \\
\dot{x}_h(\theta) &= R\dot{\theta}\cos\theta \\
\dot{y}_h(\theta) &= R\dot{\theta}\sin\theta
\end{aligned}$$

and the particle's coordinates are

$$\begin{aligned}
x_p(\theta,\psi) &= x_h + R\sin\psi \\
&= R\left[\sin\theta + \sin\psi\right] \\
y_p(\theta,\psi) &= y_h - R\cos\psi \\
&= -R\left[\cos\theta + \cos\psi\right] \\
\dot{x}_p(\theta,\psi) &= \dot{x}_h + R\dot{\psi}\cos\psi \\
&= R\left(\dot{\theta}\cos\theta + \dot{\psi}\cos\psi\right) \\
\dot{y}_h(\theta,\psi) &= \dot{y}_h + R\dot{\psi}\sin\psi \\
&= R\left(\dot{\theta}\sin\theta + \dot{\psi}\sin\psi\right).
\end{aligned}$$

So then the potential energy is

$$\begin{aligned}
U' &= Mgy_h + Mgy_p = -MgR\cos\theta - MgR\left[\cos\theta + \cos\psi\right] \\
&= -MgR\left[2\cos\theta + \cos\psi\right] \\
&\approx -MgR\left[2 - \theta^2 + 1 - \frac{\psi^2}{2}\right] \\
&= -\frac{MgR}{2}\left[6 - 2\theta^2 - \psi^2\right] \\
U &= \frac{MgR}{2}\left[2\theta^2 + \psi^2\right].
\end{aligned}$$

Therefore,

$$\boxed{\tilde{A} = MgR\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}. \tag{16}$$

Likewise, the kinetic energy is

$$\begin{aligned}
T = T_h + T_p &= \frac{I}{2}\dot{\theta}^2 + \frac{M}{2}\left(\dot{x}_p^2 + \dot{y}_p^2\right) \\
&= MR^2\dot{\theta}^2 + \frac{M}{2}\left[R^2\left(\dot{\theta}\cos\theta + \dot{\psi}\cos\psi\right)^2 + R^2\left(\dot{\theta}\sin\theta + \dot{\psi}\sin\psi\right)^2\right] \\
&\approx MR^2\dot{\theta}^2 + \frac{MR^2}{2}\left[\left(\dot{\theta} + \dot{\psi}\right)^2 + (0+0)^2\right] \\
&= \frac{MR^2}{2}\left[3\dot{\theta}^2 + 2\dot{\theta}\dot{\psi} + \dot{\psi}^2\right].
\end{aligned}$$

Therefore,

$$\boxed{\tilde{m} = MR^2\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}}. \tag{17}$$

Using $\tilde{A}$ and $\tilde{m}$ we can find the eigenfrequencies to be

$$\begin{aligned}
0 &= \left|\tilde{A} - \omega^2 \tilde{m}\right| \\
&= \left| MgR \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \omega^2 MR^2 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \right| \\
&= MR \left| \begin{pmatrix} 2g - 3R\omega^2 & -R\omega^2 \\ -R\omega^2 & g - R\omega^2 \end{pmatrix} \right| \\
&= \left(2g - 3R\omega^2\right)\left(g - R\omega^2\right) - R^2\omega^4 \\
&= 2g^2 - 3gR\omega^2 - 2gR\omega^2 + 3R^2\omega^4 - R^2\omega^4 \\
&= 2g^2 - 5gR\omega^2 + 2R^2\omega^4.
\end{aligned}$$

So then

$$\begin{aligned}
\omega^2 &= \frac{5gR \pm \sqrt{25g^2R^2 - 4 \times 2g^2 \times 2R^2}}{4R^2} \\
&= \frac{5gR \pm 3gR}{4R^2}
\end{aligned}$$

$$\boxed{\omega^2 = \begin{cases} \frac{2g}{R} \\ \frac{g}{2R} \end{cases}}. \tag{18}$$

So the eigenvectors

$$\begin{aligned}
0 &= \left[MgR \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \omega^2 MR^2 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \right] \cdot \vec{a} \\
&= MR \begin{pmatrix} 2g - 3\omega^2 R & -\omega^2 R \\ -\omega^2 R & g - \omega^2 R \end{pmatrix} \cdot \vec{a}.
\end{aligned}$$

For $\omega^2 = 2g/R$ this leads to

$$\begin{aligned}
0 &= MR \begin{pmatrix} 2g - 6g & -2g \\ -2g & g - 2g \end{pmatrix} \cdot \vec{a}_1 \\
&= \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \cdot \vec{a}_1 \\
&= 4a_{1,1} + 2a_{1,2} + 2a_{1,1} + a_{1,2} \\
&= 2a_{1,1} + a_{1,2}
\end{aligned}$$

$$\boxed{\vec{a}_1 \propto \begin{pmatrix} 1 \\ -2 \end{pmatrix}} \tag{19}$$

and for $\omega^2 = g/2R$ we have

$$\begin{aligned}
0 &= MR \begin{pmatrix} 2g - \frac{3}{2}g & -\frac{g}{2} \\ -\frac{g}{2} & g - \frac{g}{2} \end{pmatrix} \cdot \vec{a}_2 \\
&= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \vec{a}_2 \\
&= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \vec{a}_2 \\
&= a_{2,1} - a_{2,2}
\end{aligned}$$

$$\boxed{\vec{a}_2 \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \tag{20}$$

That means that we can get normal modes when the initial $\psi = \theta$ and when the initial $\psi = -\theta/2$.