Assignment 5

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1 Marion and Thornton Chapter 11

Dynamics of Rigid Bodies

1.1 Problem 11.26

In Euler coordinates the angular velocity of some rigid body is

$$\vec{\omega} = \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \tag{1}$$

Our job is to obtain the components of the angular velocity in the body coordinate system in terms of Euler angles directly from the transformation matrix

$$\tilde{\lambda} = \tilde{\lambda}_\psi \tilde{\lambda}_\theta \tilde{\lambda}_\phi \tag{2}$$

where

$$\tilde{\lambda}_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3a}$$

$$\tilde{\lambda}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{3b}$$

$$\tilde{\lambda}_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3c}$$

We can do this because we realize

$$\vec{\omega} = \vec{\dot{\phi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}}. \tag{4}$$

So all we have to do is find each of those terms in the body coordinates.

1. $\vec{\dot{\psi}}$ is along the x_3 body axis so no work is required*i.e.*

$$\vec{\dot{\psi}} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \tag{5a}$$

2. $\vec{\dot{\phi}}$ is along the x_3-body axis and so we must do a complete series of rotations:

$$\begin{aligned}
\vec{\dot{\phi}} &= \tilde{\lambda}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} \\
&= \tilde{\lambda}_\psi\tilde{\lambda}_\theta\tilde{\lambda}_\phi\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = \tilde{\lambda}_\psi\tilde{\lambda}_\theta\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} \\
&= \tilde{\lambda}_\psi\tilde{\lambda}_\theta\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = \tilde{\lambda}_\psi\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} \\
&= \tilde{\lambda}_\psi\begin{pmatrix} 0 \\ \dot{\phi}\sin\theta \\ \dot{\phi}\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \dot{\phi}\sin\theta \\ \dot{\phi}\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{pmatrix} \qquad \text{(5b)}
\end{aligned}$$

3. $\vec{\dot{\theta}}$ is along what Marion and Thorton call the line of nodes which is in their notation along the x_1''' axis which is just a λ_ψ rotation away from the body system of coordinates (see Eq. 11.96 in the text). This gives

$$\begin{aligned}
\vec{\dot{\theta}} = \lambda_\psi\begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{pmatrix} \qquad \text{(5c)}
\end{aligned}$$

Therefore, adding all three up we have the angular velocity in the body coordinates in terms of the Euler angles to be

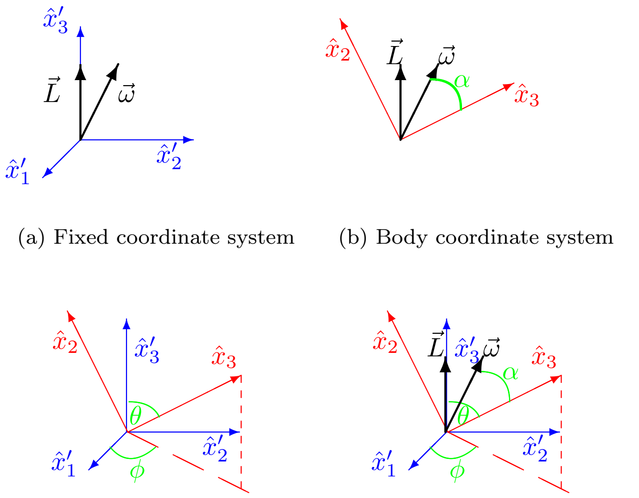
$$\begin{aligned}
\vec{\omega} &= \vec{\dot{\phi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}} \\
&= \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{pmatrix} + \begin{pmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \\
&= \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix} \qquad \text{(6)}
\end{aligned}$$

Notice that the second way to phrase this is that $\dot{\psi}$ is only a single rotation $\tilde{\lambda}_\psi$ away from body coordinates, that $\dot{\theta}$ is two rotations $(\tilde{\lambda}_\psi\tilde{\lambda}_\theta)$ away and that $\dot{\phi}$ is all three rotations away or must be transformed by $\tilde{\lambda} = \tilde{\lambda}_\psi\tilde{\lambda}_\theta\tilde{\lambda}_\phi$ which means

$$\vec{\omega} = \tilde{\lambda}_\psi\tilde{\lambda}_\theta\tilde{\lambda}_\phi\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} + \tilde{\lambda}_\psi\tilde{\lambda}_\theta\begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \tilde{\lambda}_\psi\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}.$$

These don't require as much interpretation as the first way but do require one more matrix multiplication.

1.2 Problem 11.27



(a) Fixed coordinate system (b) Body coordinate system

(c) Fixed and body coordinates. (d) Complete picture.

Figure 1: Angular momentum and angular velocity in the body (red) and fixed (blue) coordinate systems.

A symmetric body is in motion but isn't acted on by any forces or torques. The angular momentum $\vec{L}$ is along the $\hat{x}'_3$ axis in the fixed coordinate system as seen in Fig. 1a . We let $\hat{x}_3$ be the symmetry axis in the body system of coordinates. We let the angular velocity and momentum to both be in the $\hat{x}_3$-$\hat{x}_2$ plane and let the angular velocity $\vec{\omega}$ be some angle α from $\hat{x}_3$ in this plane as seen in Fig. 1b .

Because $\vec{L}$ is along $\hat{x}'_3$ and in the $\hat{x}_3$-$\hat{x}_2$ plane, we can say that the body coordinates and the fixed coordinates are related by the angle θ from the body axis x_3 to the fixed axis $\hat{x}'_3$ (see Fig. 1c).

Also, since $\vec{L}$ is in the $\hat{x}_3$-$\hat{x}_2$ plane **and** along $\hat{x}'_3$ it follows that $\hat{x}'_3$ is in the $\hat{x}_3$-$\hat{x}_2$ plane. Therefore, the $\hat{x}_3$-$\hat{x}_2$ plane can be projected down onto the $\hat{x}'_1$-$\hat{x}'_2$ plane and at any instant can be described by and angle ϕ from $\hat{x}'_1$. This is shown in Fig. 1c . The angles θ and ϕ are Euler angles.

This is all put together into Fig. 1d .

Question: What is the angular velocity of the symmetry axis $\hat{x}_3$ about $\vec{L}$ in terms of I_1, I_3 and α?

We already defined the angle of $\hat{x}_3$ about $\hat{x}'_3$ as ϕ. Therefore since $\vec{L}$ lies along $\hat{x}'_3$ we know that the angular velocity of $\hat{x}_3$ about $\vec{L}$ is $\dot{\phi}$.

In the fixed coordinates, we know $\vec{L} = L\hat{x}'_3$ but looking at Fig. 1d we can find

$$\vec{L} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = L \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix} \tag{7}$$

in the body coordinates.

But by Fig. 1b we also know that the angular velocity in the body coordinates (through α)

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \omega \begin{pmatrix} 0 \\ \sin\alpha \\ \cos\alpha \end{pmatrix} \tag{8}$$

can give the angular momentum to be

$$\vec{L} = \tilde{I} \cdot \vec{\omega} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} = \omega \begin{pmatrix} 0 \\ I_2 \sin\alpha \\ I_3 \cos\alpha \end{pmatrix} \tag{9}$$

where I_i are the moments of inertia in the body coordinates (or the principal moments of inertia). Equating Eq. (7) and Eq. (9) relates α to the Euler angle through the principle moments of inertia:

$$L\sin\theta = \omega I_2 \sin\alpha$$
$$L\cos\theta = \omega I_3 \cos\alpha$$

and dividing these and noting that $I_1 = I_2$ by symmetry gives

$$\boxed{\tan\theta = \frac{I_1}{I_3}\tan\alpha} \tag{10}$$

We will use Eq. (10) to give $\dot{\phi}$ in terms of I_1, I_3 and α instead of θ.

In Eq. (8) we gave the angular velocity in the body coordinates but we can give $\vec{\omega}$ in the fixed coordinates through the Euler angles. This was the result of the last question (Eq. (6)):

$$\vec{\omega} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

We can arbitrarily pick $\psi = 0$ since ψ is not altered ever in this problem which reduces $\vec{\omega}$ to

$$\vec{\omega} = \begin{pmatrix} \dot{\theta} \\ \dot{\phi}\sin\theta \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix} \tag{11}$$

Comparing ω_1 in Eq. (11) and Eq. (8) we see that

$$\boxed{\dot{\theta} = 0} \tag{12}$$

which we knew because α is constant.

Comparing ω_2 in Eq. (11) and Eq. (8) tells us that the angular velocity of procession is

$$\begin{aligned} \dot{\phi}\sin\theta &= \omega\sin\alpha \\ \dot{\phi} &= \omega\frac{\sin\alpha}{\sin\theta}. \end{aligned}$$

This may not seem helpful but notice from the common identity

$$\begin{aligned} 1 &= \cos^2\theta + \sin^2\theta \\ \frac{1}{\sin^2\theta} &= 1 + \frac{\cos^2\theta}{\sin^2\theta} \\ &= 1 + \frac{1}{\tan^2\theta}. \end{aligned}$$

Therefore using Eq. (10) , we have an angular velocity of

$$\begin{aligned} \dot{\phi} &= \omega\frac{\sin\alpha}{\sin\theta} \\ &= \omega\sin\alpha\sqrt{1 + \frac{1}{\tan^2\theta}} \\ &= \omega\sin\alpha\sqrt{1 + \left(\frac{1}{\frac{I_1}{I_3}\tan\alpha}\right)^2} \\ &= \omega\sin\alpha\left[1 + \left(\frac{I_3}{I_1}\cot\alpha\right)^2\right]^{1/2} \\ &= \omega\left[\sin^2\alpha + \left(\frac{I_3}{I_1}\right)^2\cos^2\alpha\right]^{1/2} \end{aligned}$$

$$\boxed{\dot{\phi} = \omega\cos\alpha\left[\tan^2\alpha + \frac{I_3^2}{I_1^2}\right]^{1/2}}. \tag{13}$$