Assignment 3

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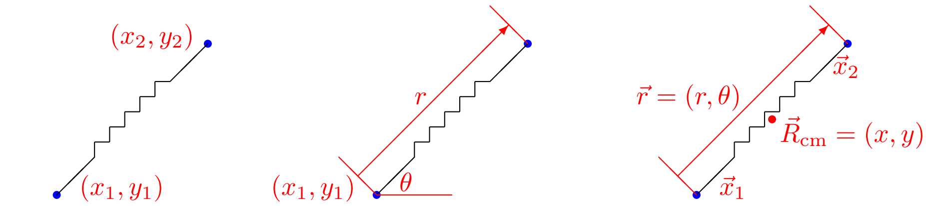
1 Marion and Thornton Chapter 7

Hamilton's Principle - Lagrangian and Hamiltonian dynamics.

1.1 Problem 7.27



(a) Cartesian Coordinates (b) Polar Coordinates (c) Centre of Mass Coordinates

Figure 1: Two particles connected by a spring.

Consider two particles of masses m_1 and m_2 attached by a spring with spring constant k and of equalibrium length b.

1.1.1 Problem 7.27.a

First let's determine the Lagrangian in pure cartesian coordiantes as shown in Fig. 1a :

$$U = \frac{k}{2}\left[\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} - b\right]^2 \tag{1a}$$

$$T = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) \tag{1b}$$

$$\mathcal{L} = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) - \frac{k}{2}\left[\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} - b\right]^2 . \tag{1c}$$

Now that's a touch silly because we know that (x_2, y_2) is related to (x_1, y_1) through r and the orientation. Although moving to these polar coordinates (see Fig. 1b) doesn't reduce the number of variables it is much more elegant, for instance check out the potential U.

So then in terms of the length r, the orientation θ and the cartesian coordinates of the first mass (now $(x_1, y_1) \to (x, y)$), the coordinates of the second are

$$
\begin{aligned}
x_2 &= x + r\cos\theta \\
y_2 &= y + r\sin\theta \\
\dot{x}_2 &= \dot{x} + \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\
\dot{y}_2 &= \dot{y} + \dot{r}\sin\theta + r\dot{\theta}\cos\theta
\end{aligned}
$$

which means that the Lagrangian can be writen

$$
U = \frac{k}{2}(r-b)^2 \tag{2a}
$$

$$
\begin{aligned}
T = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\Big[\dot{x}_1^2 + \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta \\
+2\dot{x}_1\dot{r}\cos\theta - 2r\dot{x}_1\dot{\theta}\sin\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\Big] \\
+ \frac{m_2}{2}\Big[\dot{y}_1^2 + \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta \\
+2\dot{y}_1\dot{r}\sin\theta + 2r\dot{y}_1\dot{\theta}\cos\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\Big] \\
= \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{r}^2 + r^2\dot{\theta}^2\right) \\
+ m_2\dot{r}\left(\dot{x}_1\cos\theta + \dot{y}_1\sin\theta\right) + m_2 r\dot{\theta}\left(\dot{y}_1\cos\theta - \dot{x}_1\sin\theta\right)
\end{aligned} \tag{2b}
$$

$$
\begin{aligned}
\mathcal{L} = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{r}^2 + r^2\dot{\theta}^2\right) \\
+ m_2\dot{r}\left(\dot{x}_1\cos\theta + \dot{y}_1\sin\theta\right) + m_2 r\dot{\theta}\left(\dot{y}_1\cos\theta - \dot{x}_1\sin\theta\right) - \frac{k}{2}(r-b)^2 .
\end{aligned} \tag{2c}
$$

In polar coordinates U cleaned up but T was still pretty ugly. So we do a third option for coordinates. We do centre of mass coordinates as shown in Fig. 1c . This takes a bit of work but the results are satisfying. First, for convinience sake, we use vectors to start so the vector from the centre of mass $\vec{R}_{\text{cm}}$ to each of the masses at $\vec{x}_1$ and $\vec{x}_2$ are

$$
\begin{aligned}
\vec{X}_1 &= \vec{x}_1 - \vec{R}_{\text{cm}} \\
\vec{X}_2 &= \vec{x}_2 - \vec{R}_{\text{cm}}.
\end{aligned}
$$

Furthermore, we know the definition of $\vec{R}_{\text{cm}}$ and that $\vec{x}_1$ and $\vec{x}_2$ are related through $\vec{r}$ *i.e.*

$$
\vec{r} = \vec{x}_2 - \vec{x}_1 \tag{3a}
$$

$$
\vec{R}_{\text{cm}} = \frac{\sum m_i\vec{x}_i}{\sum m_i} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}. \tag{3b}
$$

Therefore combining Eq. (3a) and Eq. (3b) , we can write

$$\begin{aligned}
\vec{X}_1 = \vec{x_1} - \vec{R}_{\rm cm} &= \left[\vec{R}_{\rm cm} - \frac{m_2}{m_1+m_2}\vec{r}\right] - \vec{R}_{\rm cm} \\
&= -\frac{1}{1+m_1/m_2}\vec{r} \\
\vec{X}_2 = \vec{x}_2 - \vec{R}_{\rm cm} &= \left[\vec{R}_{\rm cm} + \frac{m_1}{m_1+m_2}\vec{r}\right] - \vec{R}_{\rm cm} \\
&= \frac{1}{1+m_2/m_1}\vec{r} \\
\dot{\vec{X}}_1 &= -\frac{\dot{\vec{r}}}{1+m_1/m_2} \\
\dot{\vec{X}}_2 &= \frac{\dot{\vec{r}}}{1+m_2/m_1}
\end{aligned}$$

So the Lagrangian is

$$U = \frac{k}{2}\left(r-b\right)^2 \tag{4a}$$

$$\begin{aligned}
T &= \underbrace{\frac{m_1+m_2}{2}\dot{\vec{R}}_{\rm cm}^2}_{\text{CM motion}} + \underbrace{\frac{m_1}{2}\dot{\vec{X}}_1^2 + \frac{m_2}{2}\dot{\vec{X}}_2^2}_{\text{relative motion}} \\
&= \frac{m_1+m_2}{2}\dot{\vec{R}}_{\rm cm}^2 + \frac{m_1}{2}\left(-\frac{\dot{\vec{r}}}{1+m_1/m_2}\right)^2 + \frac{m_2}{2}\left(\frac{\dot{\vec{r}}}{1+m_2/m_1}\right)^2 \\
&= \frac{m_1+m_2}{2}\dot{\vec{R}}_{\rm cm}^2 + \frac{m_1 m_2}{m_1+m_2}\frac{\dot{\vec{r}}^2}{2} \\
&= \frac{M}{2}\dot{\vec{R}}_{\rm cm}^2 + \frac{\mu}{2}\dot{\vec{r}}^2
\end{aligned} \tag{4b}$$

$$\begin{aligned}
\mathcal{L} &= \frac{M}{2}\dot{\vec{R}}_{\rm cm}^2 + \frac{\mu}{2}\dot{\vec{r}}^2 - \frac{k}{2}\left(r-b\right)^2 \\
&= \frac{M}{2}\left(\dot{x}^2+\dot{y}^2\right) + \frac{\mu}{2}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)^2 - \frac{k}{2}\left(r-b\right)^2 \\
&= \frac{M}{2}\left(\dot{x}^2+\dot{y}^2\right) + \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{2}\left(r-b\right)^2
\end{aligned} \tag{4c}$$

where $M = m_1 + m_2$ is the total mass, $\mu = m_1 m_2/M$ is the reduced mass , $\vec{R}_{\rm cm} = (x, y)$ and $\vec{r} = (r, \theta)$.

Now then the Lagrange equations of motion are

Lagrange Equation for x

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\
&= -\frac{d}{dt}\left[M\dot{x}\right]
\end{aligned}$$

$$\boxed{\ddot{x} = 0} \tag{5a}$$

Lagrange Equation for y

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \\ &= -\frac{d}{dt}\left[M\dot{y}\right] \end{aligned}$$

$$\boxed{\ddot{y} = 0} \tag{5b}$$

Lagrange Equation for θ

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \\ &= -\frac{d}{dt}\left[\mu r^2 \dot{\theta}\right] = -\mu\left[r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right] \end{aligned}$$

$$\boxed{\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} = 0} \tag{5c}$$

Lagrange Equation for r

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\ &= \mu r\dot{\theta}^2 - k\left(r - b\right) - \frac{d}{dt}\mu\dot{r} \end{aligned}$$

$$\boxed{\mu\ddot{r} - \mu r\dot{\theta}^2 + k\left(r - b\right) = 0} \tag{5d}$$

Please note that the cartesian coordinates system have ugly Lagrange equations:

$$\ddot{x}_1 = \frac{k}{m_1}\left(x_2 - x_1\right) - b\frac{k}{m_1}\frac{x_2 - x_1}{\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}} \tag{6a}$$

$$\ddot{x}_2 = -\frac{k}{m_2}\left(x_2 - x_1\right) + b\frac{k}{m_2}\frac{x_2 - x_1}{\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}} \tag{6b}$$

$$\ddot{y}_1 = \frac{k}{m_1}\left(y_2 - y_1\right) - b\frac{k}{m_1}\frac{y_2 - y_1}{\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}} \tag{6c}$$

$$\ddot{y}_2 = -\frac{k}{m_2}\left(y_2 - y_1\right) + b\frac{k}{m_2}\frac{y_2 - y_1}{\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}} \tag{6d}$$

and that the polar coordinates have Lagrange equations:

$$\text{const} = m_1\dot{x}_1 + m_2\dot{x}_2 \tag{7a}$$

$$\text{const} = m_1\dot{y}_1 + m_2\dot{y}_2 \tag{7b}$$

$$0 = \ddot{r} + \ddot{x}\cos\theta + \ddot{y}\sin\theta - r\dot{\theta}^2 + \frac{k}{m_2}\left(r - b\right) \tag{7c}$$

$$0 = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + r\ddot{y}_1\cos\theta - r\ddot{x}\sin\theta. \tag{7d}$$

1.1.2 Problem 7.27.b

What are the generalized momenta that are associated with the cyclic coordinates?

In cartesian coordinates, none of the coordinates are cyclic. The generalized momenta are

$$
\begin{aligned}
p_{x_1} &= m_1 \dot{x}_1 && \text{(8a)}\\
p_{x_2} &= m_2 \dot{x}_2 && \text{(8b)}\\
p_{y_1} &= m_1 \dot{y}_1 && \text{(8c)}\\
p_{y_2} &= m_2 \dot{y}_2. && \text{(8d)}
\end{aligned}
$$

In Eq. (7) , we see clearly from the polar coordinates of Fig. 1b that

$$
\begin{aligned}
m_1 \dot{x}_1 + m_2 \dot{x}_2 &= p_x && \text{(9a)}\\
m_1 \dot{y}_1 + m_2 \dot{y}_2 &= p_y && \text{(9b)}
\end{aligned}
$$

since those are the ones for which $\partial \mathcal{L}/\partial q = 0$. Of course, there are other generalized momenta but since they aren't constant, they are not cyclic. Notice that in these coordinates, the cyclic generalized momenta are just the total linear momentum. The non-cyclic ones are

$$
\begin{aligned}
p_r &= m_2 \dot{r} + m_2 \left(\dot{x}_1 \cos\theta + \dot{y}_1 \sin\theta\right) && \text{(9c)}\\
p_\theta &= m_2 r^2 \dot{\theta} + m_2 r \left(\dot{y}_1 \cos\theta - \dot{x}_1 \sin\theta\right) && \text{(9d)}
\end{aligned}
$$

For our centre of mass coordinates, we have three coordinates for which $\partial \mathcal{L}/\partial q = 0$. They were x, y and θ (r wasn't cyclic)

$$
\begin{aligned}
M\dot{x} &= p_x && \text{(10a)}\\
M\dot{y} &= p_y && \text{(10b)}\\
\mu r^2 \dot{\theta} &= p_\theta. && \text{(10c)}
\end{aligned}
$$

This is even better than above. They are almost as simple as the cartesian ones but now we clearly see angular momentum and linear momentum are both conserved too.

The non-cyclic momentum is

$$
\mu \dot{r} = p_r \qquad \text{(10d)}
$$

1.1.3 Problem 7.27.c

Find Hamilton's equations of motion.

The Hamiltonian is (either from the energy or the definition)

$$
\begin{aligned}
\mathcal{H} &= \sum_j p_j \dot{q}_j - \mathcal{L} = \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \dot{q}_j - \mathcal{L}\\
&= T + U\\
&= \frac{M}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{2}(r-b)^2\\
&= \frac{1}{2M}\left(p_x^2 + p_y^2\right) + \frac{1}{2\mu}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + \frac{k}{2}(r-b)^2. && \text{(11)}
\end{aligned}
$$

From this point the 8 Hamilton Equations are almost trivial:

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{M} \tag{12a}$$

$$\dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = 0 \tag{12b}$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{M} \tag{12c}$$

$$\dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = 0 \tag{12d}$$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{\mu r^2} \tag{12e}$$

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = 0 \tag{12f}$$

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{\mu} \tag{12g}$$

$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\theta^2}{\mu r^3} - k\left(r - b\right) \tag{12h}$$

The 8 Hamilton Equations for the polar coordinate system is way more annoying to solve for. The solution is

$$\dot{x}_1 = \frac{\partial \mathcal{H}}{\partial p_{x_1}} = \frac{1}{m_1}\left[p_{x_1} - p_r \cos\theta + \frac{p_\theta}{r}\sin\theta\right] \tag{13a}$$

$$\dot{y}_1 = \frac{\partial \mathcal{H}}{\partial p_{y_1}} = \frac{1}{m_1}\left[p_{y_1} - p_r \sin\theta - \frac{p_\theta}{r}\cos\theta\right] \tag{13b}$$

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{1}{m_1}\left[\frac{m_1 + m_2}{m_2} p_r - p_{x_1}\cos\theta - p_{y_1}\sin\theta\right] \tag{13c}$$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{1}{r m_1}\left[\frac{m_1 + m_2}{r m_2} p_\theta + p_{x_1}\sin\theta - p_{y_1}\cos\theta\right] \tag{13d}$$

$$\dot{p}_{x_1} = -\frac{\partial \mathcal{H}}{\partial x_1} = 0 \tag{13e}$$

$$\dot{p}_{y_1} = -\frac{\partial \mathcal{H}}{\partial y_1} = 0 \tag{13f}$$

$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{m_1 + m_2}{m_1 m_2}\frac{p_\theta^2}{r^3} + \frac{p_\theta}{r^2 m_1}\left(p_{x_1}\sin\theta - p_{y_1}\cos\theta\right) - k\left(r - b\right) \tag{13g}$$

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = \frac{p_r}{m_1}\left(-p_{x_1}\sin\theta + p_{y_1}\cos\theta\right) - \frac{p_\theta}{r m_1}\left(p_{x_1}\cos\theta + p_{y_1}\sin\theta\right). \tag{13h}$$

You can see now that our little bit of extra work to go from polar to centre of mass coordinates ensures far less work at this point.

That being said, their not even that bad for cartesian coordinates

$$\mathcal{H} = \frac{p_{x_1}^2}{2m_1} + \frac{p_{y_1}^2}{2m_1} + \frac{p_{x_2}^2}{2m_2} + \frac{p_{y_2}^2}{2m_2} + \frac{k}{2}\left[\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + b\right] - kb\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

$$\dot{x}_1 = \frac{\partial \mathcal{H}}{\partial p_{x_1}} = \frac{p_{x_1}}{m_1} \tag{14a}$$

$$\dot{x}_2 = \frac{\partial \mathcal{H}}{\partial p_{x_2}} = \frac{p_{x_2}}{m_2} \tag{14b}$$

$$\dot{y}_1 = \frac{\partial \mathcal{H}}{\partial p_{y_1}} = \frac{p_{y_1}}{m_1} \tag{14c}$$

$$\dot{y}_2 = \frac{\partial \mathcal{H}}{\partial p_{y_2}} = \frac{p_{y_2}}{m_2} \tag{14d}$$

$$\dot{p}_{x_1} = -\frac{\partial \mathcal{H}}{\partial x_1} = k\left[x_2 - x_1 - b\frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}\right] \tag{14e}$$

$$\dot{p}_{x_2} = -\frac{\partial \mathcal{H}}{\partial x_2} = -k\left[x_2 - x_1 - b\frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}\right] \tag{14f}$$

$$\dot{p}_{y_1} = -\frac{\partial \mathcal{H}}{\partial y_1} = k\left[y_2 - y_1 - b\frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}\right] \tag{14g}$$

$$\dot{p}_{y_2} = -\frac{\partial \mathcal{H}}{\partial y_2} = -k\left[y_2 - y_1 - b\frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}\right] \tag{14h}$$

1.2 Problem 7.29

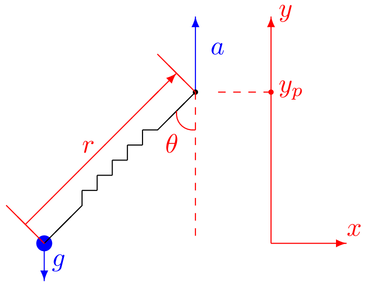


Figure 2: Particle on a pendulum spring being pulled up with an acceleration a.

Consider a particle of mass m swinging on a spring of equalibrium length b and with a spring constant of k. Also imagine that the massless pivot-point is being moved upward (against gravity) at a constant acceleration of a.

1.2.1 Problem 7.29.a

We can give the energy in terms of the distance of the mass from the pivot (r) and the angle from downward (θ) as shown in Fig. 2 . If we choose our origin at the initial position of the pivot *i.e.* $(x_{p,0}, y_{p,0}) = (0, 0,)$ and consider a reference frame in which the initial velocity of the pivot was zero $v_0 = 0$ then the position and velocity of the mass at any time t are

$$\begin{aligned} x &= x_p(t) + r\sin\theta = r\sin\theta \\ y &= y_p(t) + r\cos\theta = y_0 - v_0 t + \frac{1}{2}at^2 - r\cos\theta \\ &= \frac{1}{2}at^2 - r\cos\theta \\ \dot{x} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ \dot{y} &= at - \dot{r}\cos\theta + r\dot{\theta}\sin\theta. \end{aligned}$$

From these we can find the Lagrangian $\mathcal{L}(r, \theta, \dot{r}, \dot{\theta}, t)$:

$$\begin{aligned} U &= mgy + \frac{1}{2}k(r-b)^2 \\ &= mg\frac{at^2}{2} - mgr\cos\theta + \frac{k}{2}(r-b)^2 \qquad (15a) \\ T &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \\ &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + a^2t^2 + 2at\left[r\dot{\theta}\sin\theta - \dot{r}\cos\theta\right]\right) \qquad (15b) \\ \mathcal{L} &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + a^2t^2 + 2at\left[r\dot{\theta}\sin\theta - \dot{r}\cos\theta\right]\right) \\ &\qquad - mg\left(\frac{1}{2}at^2 - r\cos\theta\right) - \frac{k}{2}(r-b)^2 \qquad (15c) \end{aligned}$$

Now then the Lagrange equations of motion are

Lagrange Equation for r

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} \\ &= mr\dot{\theta}^2 + mat\dot{\theta}\sin\theta + mg\cos\theta - k(r-b) - \frac{d}{dt}\left[m\dot{r} - mat\cos\theta\right] \\ &= mr\dot{\theta}^2 + mg\cos\theta - k(r-b) - m\ddot{r} + ma\cos\theta \end{aligned}$$

$$\boxed{\ddot{r} - r\dot{\theta}^2 - (g+a)\cos\theta + \frac{k}{m}(r-b) = 0} \qquad (16a)$$

Lagrange Equation for θ

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \\ &= mat\dot{r}\sin\theta + matr\dot{\theta}\cos\theta - mgr\sin\theta - \frac{d}{dt}\left[mr^2\dot{\theta} + matr\sin\theta\right] \\ &= -mgr\sin\theta - 2mr\dot{r}\dot{\theta} - mr^2\ddot{\theta} - mar\sin\theta \end{aligned}$$

$$\boxed{r\ddot{\theta} + 2\dot{r}\dot{\theta} + (g+a)\sin\theta = 0} \qquad (16b)$$

1.2.2 Problem 7.29.b

We can now find the Hamiltonian since the Lagrangian gives the generalized momenta to be

$$\begin{aligned} p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} - mat\cos\theta \\ p_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} + matr\sin\theta \\ \dot{r} &= \frac{p_r}{m} + at\cos\theta \\ \dot{\theta} &= \frac{p_\theta}{mr^2} - \frac{at}{r}\sin\theta \end{aligned}$$

where we were able to identify the generalized velocities too because of the simplicity of the momenta. Along with the Lagrangian, these give the Hamiltonian to be

$$\begin{aligned} \mathcal{H} &= \sum_j p_j \dot{q}_j - \mathcal{L} \\ &= p_r\left(\frac{p_r}{m} + at\cos\theta\right) + p_\theta\left(\frac{p_\theta}{mr^2} - \frac{at}{r}\sin\theta\right) - \mathcal{L} \\ &= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + p_r at\cos\theta - \frac{p_\theta at}{r}\sin\theta - \mathcal{L} \\ &= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + p_r at\cos\theta - \frac{p_\theta at}{r}\sin\theta - \left(\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - mg\frac{at^2}{2} + mgr\cos\theta - \frac{k}{2}(r-b)^2\right) \end{aligned}$$

$$\boxed{\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + p_r at\cos\theta - \frac{p_\theta}{r}at\sin\theta + mg\left(\frac{at^2}{2} - r\cos\theta\right) + \frac{k}{2}(r-b)^2} \tag{17}$$

which gives the time derivatives of the momenta to be

$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{p_\theta}{r^2}at\sin\theta + mg\cos\theta - k(r-b) \tag{18a}$$

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = p_r at\sin\theta + \frac{p_\theta}{r}at\cos\theta - mgr\sin\theta \tag{18b}$$

1.2.3 Problem 7.29.c

We can determin the period of small oscillations by allowing the following approximations to hold if $\theta \ll 1$:

$$\begin{aligned} \sin\theta &\to \theta \\ \cos\theta &\to 1 \end{aligned}$$

and keeping only second order terms

$$\begin{aligned} \dot{\theta}^2 &\approx 0 \\ \theta^2 &\approx 0 \\ \dot{r}\dot{\theta} &\approx 0. \end{aligned}$$

Under these conditions, Eq. (16b) becomes

$$\begin{aligned}
0 &= r\ddot{\theta} + 2\dot{r}\dot{\theta} + (g+a)\sin\theta \\
&\downarrow \\
0 &\approx r\ddot{\theta} + (g+a)\,\theta \\
\ddot{\theta} &\approx -\left(\frac{g+a}{r}\right)\theta \equiv -\omega_\theta^2\theta
\end{aligned}$$

which means that the period of oscillation in the pendulum is

$$\boxed{T_\theta = \frac{2\pi}{\omega_\theta} = 2\pi\sqrt{\frac{r}{g+a}}}. \tag{19}$$

Likewise, we find the period of oscillations in the spring from Eq. (16a) by

$$\begin{aligned}
0 &= \ddot{r} - r\dot{\theta}^2 - (g+a)\cos\theta + \frac{k}{m}(r-b) \\
&\downarrow \\
0 &\approx \ddot{r} - (g+a) + \frac{k}{m}(r-b) \\
&\downarrow \\
r &= A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right) + \frac{m}{k}(a+g) + b \\
&\equiv A\cos\left(\omega_r^2 t\right) + B\sin\left(\omega_r^2 t\right) + \frac{m}{k}(a+g) + b
\end{aligned}$$

which means that the period of oscillation in the spring is

$$\boxed{T_r = \frac{2\pi}{\omega_r} = 2\pi\sqrt{\frac{m}{k}}}. \tag{20}$$