Assignment 2

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1 Marion and Thornton Chapter 7

Hamilton's Principle - Lagrangian and Hamiltonian dynamics.

1.1 Problem 7.9

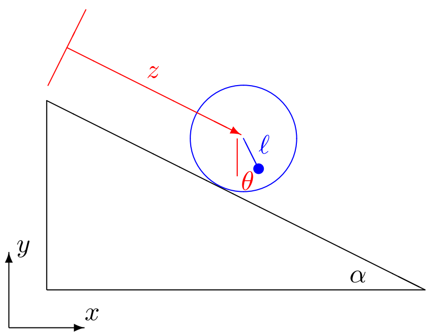


Figure 1: A disk rolling down an incline plane. From the axis of the disk hangs a simple pendulum.

Consider a disk of mass M and radius R that's rolling down a plane of angle α as shown in Fig. 1 . From the axis of this disk is a simple pendulum of length $\ell < R$ with a bob of mass m. Consider the motion of the pendulum.

We'll do this in z, θ coordinates as shown in Fig. 1 . The Cartesian coordinates in therms of z and θ are

$$\begin{aligned} x_{\text{disk}} &= z \cos \alpha \\ y_{\text{disk}} &= -z \sin \alpha \\ \dot{x}_{\text{disk}} &= \dot{z} \cos \alpha \\ \dot{y}_{\text{disk}} &= -\dot{z} \sin \alpha. \end{aligned}$$

The Cartesian coordinates of the bob are just that of a simple pendulum but in the regerence frame of the disk *i.e.*

$$\begin{aligned} x_{\text{bob}} &= x_{\text{disk}} + \ell \sin \theta = z \cos \alpha + \ell \sin \theta \\ y_{\text{bob}} &= y_{\text{disk}} - \ell \cos \theta = -z \sin \alpha - \ell \cos \theta \\ \dot{x}_{\text{bob}} &= \dot{z} \cos \alpha + \ell \dot{\theta} \cos \theta \\ \dot{y}_{\text{bob}} &= -\dot{z} \sin \alpha + \ell \dot{\theta} \sin \theta \end{aligned}$$

Also notice that the rotation angle ϕ of the disk is simply proportional to the distance travelled *i.e.*

$$\phi = \frac{z}{R}$$

If we take the moment of inertia about the centre of the disk to be

$$I = \frac{MR^2}{2}$$

then the energy and Lagrangian are then given to be

$$\begin{aligned}
U &= U_{\text{disk}} + U_{\text{bob}} = Mgy_{\text{disk}} + mgy_{\text{bob}} \\
&= -gz\,(M+m)\sin\alpha - mg\ell\cos\theta \qquad \text{(1a)} \\
T &= T_{\text{bob}} + T_{\text{kin,disk}} + T_{\text{rot,disk}} \\
&= \frac{m}{2}\left[\dot{x}_{\text{bob}}^2 + \dot{y}_{\text{bob}}^2\right] + \frac{M}{2}\left(\dot{x}_{\text{disk}}^2 + \dot{y}_{\text{disk}}^2\right) + \frac{1}{2}I\dot{\phi}^2 \\
&= \frac{m}{2}\left[\left(\dot{z}\cos\alpha + \ell\dot{\theta}\cos\theta\right)^2 + \left(-\dot{z}\sin\alpha + \ell\dot{\theta}\sin\theta\right)^2\right] \\
&\quad + \frac{M}{2}\left(\dot{z}^2\cos^2\alpha + \dot{z}^2\sin^2\alpha\right) + \frac{1}{2}\frac{MR^2}{2}\left(\frac{\dot{z}}{R}\right)^2 \\
&= \frac{\dot{z}^2}{2}(M+m) + \frac{m}{2}\ell^2\dot{\theta}^2 + m\dot{z}\dot{\theta}\ell\left(\cos\alpha\cos\theta - \sin\alpha\sin\theta\right) + \frac{M\dot{z}^2}{4} \\
&= \frac{\dot{z}^2}{2}\left(\frac{3}{2}M + m\right) + \frac{m}{2}\ell^2\dot{\theta}^2 + m\dot{z}\dot{\theta}\ell\cos(\alpha+\theta) \qquad \text{(1b)} \\
\mathcal{L} &= T - U \\
&= \frac{\dot{z}^2}{2}\left(\frac{3}{2}M + m\right) + \frac{m}{2}\ell^2\dot{\theta}^2 + m\dot{z}\dot{\theta}\ell\cos(\alpha+\theta) \\
&\quad + gz\,(M+m)\sin\alpha + mg\ell\cos\theta \qquad \text{(1c)}
\end{aligned}$$

Notice, since it's come up multiple times the kinetic energy of the disk and the bob are

$$\begin{aligned}
T_{\text{disk}} &= T_{\text{kin,disk}} + T_{\text{rot,disk}} = \frac{M}{2}\dot{z}^2 + \frac{M}{4}\dot{z}^2 \\
&= \frac{3}{4}M\dot{z}^2 \\
T_{\text{bob}} &= \frac{m}{2}\left[\dot{x}_{\text{bob}}^2 + \dot{y}_{\text{bob}}^2\right] \\
&= \frac{m}{2}\left[\dot{z}^2 + \ell^2\dot{\theta}^2 + 2\dot{z}\dot{\theta}\ell\left(\cos\alpha\cos\theta - \sin\alpha\sin\theta\right)\right] \\
&= \frac{m}{2}\left[\dot{z}^2 + \ell^2\dot{\theta}^2 + 2\dot{z}\dot{\theta}\ell\cos(\alpha+\theta)\right]
\end{aligned}$$

Lagrange Equation for z

$$\begin{aligned}
0 &= \frac{\partial\mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{z}} \\
&= Mg\sin\alpha + mg\sin\alpha - \frac{d}{dt}\left[\dot{z}\left(\frac{3}{2}M + m\right) + m\dot{\theta}\ell\cos(\alpha+\theta)\right] \\
&= (M+m)\,g\sin\alpha - \left[\ddot{z}\left(\frac{3}{2}M + m\right) + m\ell\ddot{\theta}\cos(\alpha+\theta) - m\ell\dot{\theta}^2\sin(\alpha+\theta)\right]
\end{aligned}$$

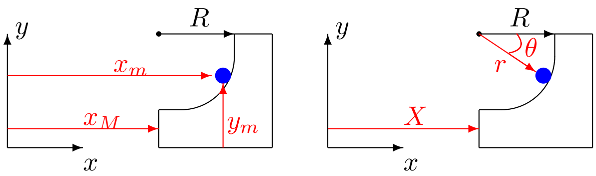
$$\boxed{\ddot{z}\left(\frac{3}{2}M+m\right)+m\ell\ddot{\theta}\cos\left(\alpha+\theta\right)-m\ell\dot{\theta}^2\sin\left(\alpha+\theta\right)-\left(M+m\right)g\sin\alpha=0} \tag{2}$$

Lagrange Equation for θ

$$\begin{aligned}
0 &= \frac{\partial\mathcal{L}}{\partial\theta}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \\
&= -m\dot{z}\dot{\theta}\ell\sin\left(\alpha+\theta\right)-mg\ell\sin\theta-\frac{d}{dt}\left[m\ell^2\dot{\theta}+m\dot{z}\ell\cos\left(\alpha+\theta\right)\right] \\
&= -m\dot{z}\dot{\theta}\ell\sin\left(\alpha+\theta\right)-mg\ell\sin\theta-\left[m\ell^2\ddot{\theta}+m\ddot{z}\ell\cos\left(\alpha+\theta\right)-m\dot{z}\ell\dot{\theta}\sin\left(\alpha+\theta\right)\right]
\end{aligned}$$

$$\boxed{\ell\ddot{\theta}+\ddot{z}\cos\left(\alpha+\theta\right)+g\sin\theta=0} \tag{3}$$

1.2 Problem 7.34



(a) Cartesian Coordinates (b) Unconstrained Coordinates

Figure 2: A particle sliding down a smooth circular wedge.

Consider a particle of mass m sliding down a smooth circular wedge of mass M. The particle (denoted by subscript m) slides - not rolls. The wedge (denoted by subscript M) slides on a smooth horizontal surface.

Of course, you can always chose whatever coordinates you want but some are more judicious than others. We will use X, r, θ from Fig. 2b . We can related these "unconstrained" coordinates to the (perhaps more intuitive) cartesian coordinates of Fig. 2a .

Wedge:

$$\begin{aligned}
x_M &= X \\
y_M &= 0
\end{aligned}$$

Particle:

$$\begin{aligned}
x_m &= X + r\cos\theta \\
y_m &= -r\sin\theta
\end{aligned}$$

$$
\begin{aligned}
\dot{x}_m &= \dot{X} + \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\
\dot{y}_m &= -\dot{r}\sin\theta - r\dot{\theta}\cos\theta
\end{aligned}
$$

Now this is really great because we can set up the energy in cartesian and then talk about the unconstrained coordinates after their set up *i.e.*

$$
\begin{aligned}
U &= mgy_m \\
&= -mgr\sin\theta \qquad \text{(4a)} \\
T &= \underbrace{\frac{1}{2}M\dot{x}_M^2}_{\text{wedge}} + \underbrace{\frac{1}{2}m\left[\dot{x}_m^2 + \dot{y}_m^2\right]}_{\text{particle}} \\
&= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\Big[\dot{X}^2 + 2\dot{X}\dot{r}\cos\theta - 2\dot{X}r\dot{\theta}\sin\theta + \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta \\
&\qquad + r^2\dot{\theta}^2\sin^2\theta + \dot{r}^2\sin^2\theta + 2\dot{r}r\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\cos^2\theta\Big] \\
&= \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m\Big[\dot{r}^2\left(\cos^2\theta + \sin^2\theta\right) \\
&\qquad + r^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right) + 2\dot{X}\dot{r}\cos\theta - 2\dot{X}\dot{\theta}r\sin\theta\Big] \\
&= \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{X}\dot{r}\cos\theta - 2\dot{X}\dot{\theta}r\sin\theta\right] \qquad \text{(4b)} \\
\mathcal{L} &= T - U \\
&= \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{X}\dot{r}\cos\theta - 2\dot{X}\dot{\theta}r\sin\theta\right] \\
&\qquad + mgr\sin\theta \qquad \text{(4c)}
\end{aligned}
$$

Now we can get the Lagrange equations for each of the two or three variables.

When we recognize that $r = R$ right off the bat then there is no need for a Lagrange multiplier λ and so we can completely solve the problem by the two Lagrange equations for θ and X.

However, if we leave r as a variable and consider a third Lagrange equation then we require a λ that connects them. By doing this we will learn about the physical process that ensures $r = R$.

Since $r = R$ is what we wish to ensure by including λ, the function f is

$$
\boxed{f(X,\theta,r) = r - R = 0}. \qquad \text{(5)}
$$

Notice,

$$
\frac{\partial f}{\partial X} = 0, \quad \frac{\partial f}{\partial \theta} = 0, \quad \frac{\partial f}{\partial r} = 1
$$

Lagrange Equation for X

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} + \lambda\frac{\partial f}{\partial X} &= 0 \\
\frac{\partial \mathcal{L}}{\partial X} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} &= 0 \qquad \text{(6a)}
\end{aligned}
$$

Lagrange Equation for θ

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} + \lambda \frac{\partial f}{\partial \theta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial X} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = 0 \tag{6b}$$

Lagrange Equation for r

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} + \lambda \frac{\partial f}{\partial r} = 0$$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} + \lambda = 0 \tag{6c}$$

We see that the equations for X and θ are no different than if we hadn't used the multiplier but we have the extra equation for r which will give us the force of constraint that ensures $r = R$ *i.e.* the force of the wedge and the particle on each other.

So now let's find the equations of motion from Eq. (6a) and Eq. (6b) (later we'll find the force of constraint from Eq. (6c)).

Lagrange Equation for X

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial X} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \\
&= 0 - \frac{d}{dt}\left[(M+m)\dot{X} + \frac{1}{2}m\left(0 + 0 + 2\dot{r}\cos\theta - 2\dot{\theta}r\sin\theta\right) + 0\right] \\
&= (M+m)\ddot{X} + m\left[\ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \ddot{\theta}r\sin\theta - \dot{\theta}\dot{r}\sin\theta - r\dot{\theta}^2\cos\theta\right] \\
&= (M+m)\ddot{X} + m\left[\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\right].
\end{aligned}$$

Applying the constraint $r = R$ gives $\ddot{r} = 0 = \dot{r}$ so that we can see

$$0 = (M+m)\ddot{X} - mR\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$$

$$\boxed{\ddot{X} = \frac{m}{M+m}R\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)} \tag{7}$$

Lagrange Equation for θ

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \\
&= 0 + \frac{1}{2}m\left[0 + 0 - 2\dot{X}\dot{r}\sin\theta - 2\dot{X}\dot{\theta}r\cos\theta\right] + mgr\cos\theta \\
&\quad - \frac{d}{dt}\left[0 + \frac{1}{2}m\left(0 + 2r^2\dot{\theta} + 0 - 2\dot{X}r\sin\theta\right) + 0\right] \\
&= -\dot{X}\dot{r}\sin\theta - \dot{X}\dot{\theta}r\cos\theta + gr\cos\theta - \frac{d}{dt}\left[r^2\dot{\theta} - \dot{X}r\sin\theta\right] \\
&= -\dot{X}\dot{r}\sin\theta - \dot{X}\dot{\theta}r\cos\theta + gr\cos\theta \\
&\quad - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} + \ddot{X}r\sin\theta + \dot{X}\dot{r}\sin\theta - \dot{X}r\dot{\theta}\cos\theta \\
&= \dot{r}\left[-\dot{X}\sin\theta - 2r\dot{\theta} + \dot{X}\sin\theta\right] + r\left[g\cos\theta - r\ddot{\theta} + \ddot{X}\sin\theta\right].
\end{aligned}$$

Again r is constant so this reduces to

$$
\begin{aligned}
0 &= 0 + R\left[g\cos\theta - R\ddot{\theta} + \ddot{X}\sin\theta\right] \\
R\ddot{\theta} &= \ddot{X}\sin\theta + g\cos\theta
\end{aligned}
$$

$$
\boxed{\ddot{\theta} = \frac{\ddot{X}\sin\theta + g\cos\theta}{R}} \tag{8}
$$

i.e. X and θ are given by two second order coupled ODEs.

Since we kept r as a variable in Eq. (4c) we are already in the perfect position to find the Lagrange multiplier λ from Eq. (6c) .

$$
\begin{aligned}
0 &= \frac{\partial\mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} + \lambda \\
&= 0 + \frac{1}{2}m\left[0 + 2r\dot{\theta}^2 + 0 - 2\dot{X}\dot{\theta}\sin\theta\right] + mg\sin\theta \\
&\quad - \frac{d}{dt}\left[0 + \frac{1}{2}m\left(2\dot{r} + 0 + 2\dot{X}\cos\theta - 0\right) + 0\right] + \lambda \\
&= r\dot{\theta}^2 - \dot{X}\dot{\theta}\sin\theta + g\sin\theta - \left(\ddot{r} + \ddot{X}\cos\theta - \dot{X}\dot{\theta}\sin\theta\right) + \frac{\lambda}{m} \\
\frac{\lambda}{m} &= -r\dot{\theta}^2 - g\sin\theta + \ddot{r} + \ddot{X}\cos\theta
\end{aligned}
$$

Now using that $r = R$ and $\dot{r} = 0 = \ddot{r}$ we can say

$$
\begin{aligned}
\lambda &= -m\left(r\dot{\theta}^2 + g\sin\theta - \ddot{X}\cos\theta\right) \qquad (9) \\
&= F_r
\end{aligned}
$$

where we have explicitly put a minus sign in to remind us that this constraining force, normal to the circular surface is centripetal and has a direction inward.

We can use our solutions for $\ddot{\theta}$ and $\ddot{X}$ to get F_r in terms of θ and $\dot{\theta}$ only by solving Eq. (7) and Eq. (8) for $\ddot{X}$.

$$
\begin{aligned}
\ddot{X} &= \frac{m}{M+m}R\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \\
&= \frac{m}{M+m}\left(\ddot{X}\sin^2\theta + g\cos\theta\sin\theta + R\dot{\theta}^2\cos\theta\right) \\
\ddot{X} - \frac{m}{M+m}\ddot{X}\sin^2\theta &= \frac{m}{M+m}\left(g\cos\theta\sin\theta + R\dot{\theta}^2\cos\theta\right) \\
\ddot{X}\left(1 - \frac{m}{M+m}\left(1-\cos^2\theta\right)\right) &= \frac{m}{M+m}\left(g\cos\theta\sin\theta + R\dot{\theta}^2\cos\theta\right) \\
\ddot{X}\left(\frac{M}{M+m} + \frac{m}{M+m}\cos^2\theta\right) &= \frac{m}{M+m}\left(g\cos\theta\sin\theta + R\dot{\theta}^2\cos\theta\right) \\
\ddot{X}\left(M + m\cos^2\theta\right) &= m\left(g\cos\theta\sin\theta + R\dot{\theta}^2\cos\theta\right)
\end{aligned}
$$

$$
\boxed{\ddot{X} = m\frac{g\cos\theta\sin\theta + R\dot{\theta}^2\cos\theta}{M + m\cos^2\theta}} \tag{10}
$$

With $\ddot{X}(\theta, \dot{\theta})$ we can give F_r to be

$$\begin{aligned}
F_r &= -m\left(R\dot{\theta}^2 + g\sin\theta - \ddot{X}\cos\theta\right) \\
&= -m\left(R\dot{\theta}^2 + g\sin\theta - m\frac{g\sin\theta\cos^2\theta + R\dot{\theta}^2\cos^2\theta}{M + m\cos^2\theta}\right) \\
&= -m\left(\frac{MR\dot{\theta}^2 + Mg\sin\theta + Rm\dot{\theta}^2\cos^2\theta + gm\sin\theta\cos^2\theta}{M + m\cos^2\theta}\right. \\
&\qquad\qquad \left. - m\frac{g\sin\theta\cos^2\theta + R\dot{\theta}^2\cos^2\theta}{M + m\cos^2\theta}\right) \\
&= -\frac{m}{M - m\cos^2\theta}\left(MR\dot{\theta}^2 + Mg\sin\theta\right)
\end{aligned}$$

$$\boxed{F_r = -\frac{mM\left(R\dot{\theta}^2 + g\sin\theta\right)}{M - m\cos^2\theta}} \tag{11}$$