Assignment 2

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1 Kadar Ch. 2 Problem 8

We have an $N \times N$ symmetric matrix, $\mathbf{M}$. The symmetry means $\mathbf{M} = \mathbf{M}^T$ and we'll say the elements of the matrix are m_{ij}. The elements are pulled from a probability density

$$p(M_{ij}) = \begin{cases} 1/2a & \text{for } -a < m_{ij} < a \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

1.1 Problem 8 a)

The characteristic function is just the fourier transform of the probability density so for each element

$$\begin{aligned} \tilde{p}(k) &= \int p(m_{ij}) e^{-ikm_{ij}} dm_{ij} \\ &= \int_{-\infty}^{a} 0 + \int_{-a}^{a} p(m_{ij}) e^{-ikm_{ij}} dm_{ij} + \int_{a}^{\infty} 0 \\ &= \int_{-a}^{a} \frac{1}{2a} e^{-ikm_{ij}} dm_{ij} \\ &= -\frac{1}{2a}\frac{1}{ik}\, e^{-ikm_{ij}} \Big|_{-a}^{a} \\ &= \frac{1}{2aik}\left[e^{ika} - e^{-ika}\right] \end{aligned}$$

$$\boxed{\tilde{p}(k) = \frac{\sin(ka)}{ka}} \tag{2}$$

1.2 Problem 8 b)

To find the probability of getting a specific trace, we could use the definition

$$\begin{aligned} p_T(T) &= \int d^N(\{m_{ii}\}) p(\{m_{ii}\}) \\ &= \int \prod^{N} dm_{ii} p(m_{ii}) \end{aligned}$$

then take

$$\tilde{p}_T = \left\langle \exp\left(-i\sum^{N} km_{ii}\right)\right\rangle$$

but that's a lot of work. Instead we should recognize

$$\begin{aligned}\tilde{p}_T(T) &= \prod_i^N \tilde{p}_i(k) \\ &= \prod_i^N \frac{\sin(ka)}{ka}\end{aligned}$$

$$\boxed{\tilde{p}_T(T) = \left[\frac{\sin(ka)}{ka}\right]^N} \tag{3}$$

2 Kadar Ch. 2 Problem 10

This problem is a practice in changing variables.

The current is a function of voltage by

$$I(V) = I_0\left[\exp\left(eV/k_{\mathrm{B}}T\right) - 1\right] \tag{4}$$

The instantaneous voltage is drawn from a gaussian distribution of zero mean and variance σ^2. We call this probability density p_V.

2.1 Problem 10 a)

What is the probability density of current, p_I?

It may not be obvious how the probability densities are related but it may be clearer to you how the probabilities are. If there is some probability P of getting some current V what is the probability of getting the corresponding current from Eq. 4. It must be the same, of course. Therefore, using the definition of probability from probability density, we see

$$\begin{aligned}P &= p_V dV = p_I dI \\ p_I &= p_V \frac{dV}{dI}.\end{aligned} \tag{5}$$

We want to give p_I as a function of current not voltage so for future use we rearrange Eq. 4 to be

$$V(I) = \frac{k_{\mathrm{B}}T}{e}\ln\left[\frac{I + I_0}{I_0}\right]. \tag{6}$$

We are now ready to find the probability density for the current:

$$\begin{aligned}
p_I &= p_V \frac{dV}{dI} \\
&= p_V \frac{d}{dI}\left(\frac{k_\mathrm{B}T}{e}\ln\left[\frac{I+I_0}{I_0}\right]\right) \\
&= \frac{k_\mathrm{B}T}{e}\frac{1}{I+I_0}p_V(V) \\
&= \frac{k_\mathrm{B}T}{e}\frac{1}{I+I_0}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{V^2}{2\sigma^2}\right)\right) \\
&= \frac{k_\mathrm{B}T}{e}\frac{1}{I+I_0}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}\left(\frac{k_\mathrm{B}T}{e}\ln\left[\frac{I+I_0}{I_0}\right]\right)^2\right)\right)
\end{aligned}$$

2.2 Problem 10 b)

What is the mean value for the current?

The average current is the current that corresponds to the average voltage. We don't want to deal with p_I since it's so ugly.

$$\begin{aligned}
< I > &= \int_{-\infty}^{\infty} I p_I dI \\
&= \int_{-\infty}^{\infty} I\left(p_V\frac{dV}{dI}\right)dI \\
&= \int_{-\infty}^{\infty} I(V)p_V dV \\
&= \int_{-\infty}^{\infty} I_0\left[\exp\left(eV/k_\mathrm{B}T\right)-1\right]p_V dV \\
&= I_0\int_{-\infty}^{\infty}\exp\left(eV/k_\mathrm{B}T\right)p_V dV - I_0\underbrace{\int_{-\infty}^{\infty}p_V dV}_{=1} \\
&= I_0\int_{-\infty}^{\infty}\exp\left(eV/k_\mathrm{B}T\right)\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{V^2}{2\sigma^2}\right)\right)dV - I_0 \\
&= \frac{I_0}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left(\frac{e}{k_\mathrm{B}T}V-\frac{1}{2\sigma^2}V^2\right)dV - I_0 \\
&= \frac{I_0}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left(\alpha V-\beta V^2\right)dV - I_0 \\
&= \frac{I_0}{\sqrt{2\pi\sigma^2}}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{\alpha^2}{4\beta}\right) - I_0 \\
&= \frac{I_0}{\sqrt{2\pi\sigma^2}}\sqrt{2\pi\sigma^2}\exp\left(\frac{2\sigma^2}{4}\left(\frac{e}{k_\mathrm{B}T}\right)^2\right) - I_0
\end{aligned}$$

$$\boxed{< I >= I_0\left[\exp\left(\frac{e\sigma}{\sqrt{2}k_\mathrm{B}T}\right)^2-1\right]} \tag{7}$$

2.3 Most Probable Current

The most probable current is the current at the maximum of the probablity density so

$$\begin{aligned}\frac{dp_I}{dI} &= 0\\ &= \frac{d}{dI}\left(\frac{k_{\mathrm{B}}T}{e}\frac{1}{I+I_0}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}\left(\frac{k_{\mathrm{B}}T}{e}\ln\left[\frac{I+I_0}{I_0}\right]\right)^2\right)\right)\right)\\ &= \frac{k_{\mathrm{B}}T}{e\sqrt{2\pi\sigma^2}}\frac{d}{dI}\left[\frac{1}{I+I_0}\exp\left(-\frac{1}{2\sigma^2}\left(\frac{k_{\mathrm{B}}T}{e}\ln\left[\frac{I+I_0}{I_0}\right]\right)^2\right)\right]\end{aligned}$$

Dropping the constants and setting $A = \frac{1}{2\sigma^2}\left(\frac{k_{\mathrm{B}}T}{e}\right)^2$ gives

$$\begin{aligned}0 &= \frac{d}{dI}\left[\frac{1}{I+I_0}\exp\left(-A\left(\ln\left[\frac{I+I_0}{I_0}\right]\right)^2\right)\right]\\ &= \frac{1}{(I+I_0)^2}\left[-2A\ln\left(\frac{I+I_0}{I_0}\right)-1\right]\exp\left(-A\left(\ln\left[\frac{I+I_0}{I_0}\right]\right)^2\right)\end{aligned}$$

Drop the exponent and the inconsequential fraction to see

$$\begin{aligned}0 &= -2A\ln\left(\frac{I+I_0}{I_0}\right)-1\\ -\frac{1}{2A} &= \ln\left(\frac{I+I_0}{I_0}\right)\\ \frac{I+I_0}{I_0} &= \exp\left(-\frac{1}{2A}\right)\\ I &= I_0\left[\exp\left(-\frac{1}{2A}\right)-1\right]\end{aligned}$$

which after all that work is

$$\boxed{I = I_0\left[\exp\left\{-\left(\frac{e\sigma}{k_{\mathrm{B}}T}\right)^2\right\}-1\right]}. \tag{8}$$

3 Sethna Ch. 5 Problem 12

We have a chain made up of n_+ steps to the right and n_- steps to the left. That means we have a total of $N = n_+ + n_-$ steps and if each link in the chain has length d then the chain's length is $L = (n_+ - n_-)\,d$.

3.1 Problem 12 a)

We can rearrange the number of steps and the length to give n_+ and n_- in terms of the other variables

$$n_+ = \frac{N}{2} + \frac{L}{2d} \tag{9}$$

$$n_- = \frac{N}{2} - \frac{L}{2d}. \tag{10}$$

The total number of configurations is

$$\begin{aligned}\Omega &= \frac{N!}{n_+!n_-!} \\ &= \frac{N!}{\left(\frac{N}{2}+\frac{L}{2d}\right)!\left(\frac{N}{2}-\frac{L}{2d}\right)!}.\end{aligned} \tag{11}$$

The entropy is just the logarithm of the number of configurations

$$\begin{aligned}S_{\text{band}} &= k_{\text{B}} \ln \Omega \\ &= k_{\text{B}} \ln \left(\frac{N!}{\left(\frac{N}{2}+\frac{L}{2d}\right)!\left(\frac{N}{2}-\frac{L}{2d}\right)!} \right)\end{aligned}$$

$$\boxed{S_{\text{band}} = k_{\text{B}} \left[\ln N! - \ln \left(\frac{N}{2} + \frac{L}{2d} \right)! - \ln \left(\frac{N}{2} - \frac{L}{2d} \right)! \right]}. \tag{12}$$

Of course, for this to be useful we will probably have to use Stirling's formula but this is the exact answer.

3.2 Problem 12 b)

We know

$$-F = \frac{dE_{\text{bath}}}{dL} \tag{13}$$

and we also know

$$\frac{1}{T} = \frac{\partial S_{\text{bath}}}{\partial E}. \tag{14}$$

Putting the two together results in

$$-\frac{F}{T} = \frac{\partial S_{\text{bath}}}{\partial L}. \tag{15}$$

The above only dealt with the entropy of the bath. But what's the change in length doing? It's maximizing the total/universal entropy. So if $S = S_{\text{bath}} + S_{\text{band}}$ then

$$\begin{aligned}0 &= \frac{\partial S}{\partial L} \\ &= \frac{\partial S_{\text{bath}}}{\partial L} + \frac{\partial S_{\text{band}}}{\partial L} \\ &= -\frac{F}{T} + \frac{\partial S_{\text{band}}}{\partial L}\end{aligned}$$

$$\boxed{\frac{\partial S_{\text{band}}}{\partial L} = \frac{F}{T}} \tag{16}$$

3.3 Problem 12 c)

To find the force in terms of the number of monomers, we just combine the results of the last two section and use Stirling's formula:

$$
\begin{aligned}
\frac{F}{T} &= \frac{\partial S_{\text{band}}}{\partial L} \\
F &= T\frac{\partial S_{\text{band}}}{\partial L} \\
&= T\frac{\partial}{\partial L}k_{\text{B}}\left[\ln N! - \ln\left(\frac{N}{2}+\frac{L}{2d}\right)! - \ln\left(\frac{N}{2}-\frac{L}{2d}\right)!\right] \\
&= k_{\text{B}}T\frac{\partial}{\partial L}\left[-\left(\frac{N}{2}+\frac{L}{2d}\right)\ln\left(\frac{N}{2}+\frac{L}{2d}\right)+\left(\frac{N}{2}+\frac{L}{2d}\right)-\left(\frac{N}{2}-\frac{L}{2d}\right)\ln\left(\frac{N}{2}-\frac{L}{2d}\right)+\left(\frac{N}{2}-\frac{L}{2d}\right)\right] \\
&= k_{\text{B}}T\left[-\frac{1}{2d}\ln\left(\frac{N}{2}+\frac{L}{2d}\right)-1+\frac{1}{2d}+\frac{1}{2d}\ln\left(\frac{N}{2}-\frac{L}{2d}\right)+1-\frac{1}{2d}\right] \\
&= k_{\text{B}}T\left[-\frac{1}{2d}\ln\left(\frac{N}{2}+\frac{L}{2d}\right)+\frac{1}{2d}\ln\left(\frac{N}{2}-\frac{L}{2d}\right)\right] \\
&= -\frac{k_{\text{B}}T}{2d}\left[\ln\left(\frac{\frac{N}{2}+\frac{L}{2d}}{\frac{N}{2}-\frac{L}{2d}}\right)\right]
\end{aligned}
$$

$$
\boxed{F = -\frac{k_{\text{B}}T}{2d}\ln\left(\frac{Nd+L}{Nd-L}\right)} \tag{17}
$$

To see the spring constant, we rewrite the term inside the logarithm and then expand it (assume $L \ll Nd$ *i.e.* the instantaneous length L is far less than it's contour/full length Nd).

$$
\begin{aligned}
F &= -\frac{k_{\text{B}}T}{2d}\ln\left(1+2\frac{L}{Nd}\right) \\
&= -\frac{k_{\text{B}}T}{2d}\left(2\frac{L}{Nd}+\ldots\right) \\
&= -\frac{k_{\text{B}}T}{Nd^2}L.
\end{aligned}
$$

So we identify the spring constant to be

$$
\boxed{K = \frac{k_{\text{B}}T}{Nd^2}}. \tag{18}
$$

3.4 Problem 12 d)

What happens if we heat the elastic band while it's under tension?

The change in length with temperature is

$$
\left.\frac{\partial L}{\partial T}\right|_F = -\left.\frac{\partial L}{\partial F}\right|_T \left.\frac{\partial F}{\partial T}\right|_L . \tag{19}
$$

The term $\left.\frac{\partial F}{\partial T}\right|_L$ is a Maxwell equation:

$$
\left.\frac{\partial F}{\partial T}\right|_L = \left.\frac{\partial S}{\partial L}\right|_T = \frac{F}{T}
$$

which we found earlier. And the other term is either by definition (if you remember these kinds of things) or from the last part related to the spring constant

$$\left.\frac{\partial L}{\partial F}\right|_T = \frac{1}{K}$$

Now Eq. 19 becomes

$$\begin{aligned}\left.\frac{\partial L}{\partial T}\right|_F &= -\left.\frac{\partial L}{\partial F}\right|_T \left.\frac{\partial F}{\partial T}\right|_L \\ &= -\frac{1}{K}\frac{F}{T} \\ &= -\frac{1}{K}\frac{KL}{T} \\ &= -\frac{L}{T}.\end{aligned}$$

Since both length and temperature must be positive, the rubber band under tension $\boxed{\text{contracts}}$ when heated.

4 Sethna Ch. 5 Problem 15

The A'bç! language is made up of three letters/sounds They have probable

Sounds	
hoot	A
slap	B
click	C

occurances of $p_A = p_B = 1/4$ and $p_C = 1/2$.

4.1 Problem 15 a)

What is the Shannon entropy per letter transmitted (or Shannon entropy rate)?

The entropy per sound is

$$\begin{aligned} S &= -k_s \sum_{\text{A'bç!}} p(i) \ln p(i) \\ &= -k_s \left[\frac{1}{4}\ln\left(\frac{1}{4}\right) + \frac{1}{4}\ln\left(\frac{1}{4}\right) + \frac{1}{2}\ln\left(\frac{1}{2}\right)\right] \\ &= -k_s \left[-\frac{1}{2}\ln 2 - \frac{1}{2}\ln 2 - \frac{1}{2}\ln 2\right] \\ &= k_s \frac{3}{2}\ln 2 \end{aligned}$$

4.2 Problem 15 b)

Show that a communication transmitting bits can transmit no more than one unit of Shannon entropy per bit.

We are looking for the maximum Shannon entropy per bit:

$$\frac{\partial}{\partial p_j} S_{SH} = 0$$

The Shannon entropy will depend on the probability distribution which (as always) will be subject to the constraint $\sum_i p_i = 1$.

Maximize $S_{SH} = -k_s \sum_i p_i \ln p_i = \sum_i p_i \log_2 p_i$ for units of bits, as in the other question.

To implement the Lagrange multiplier method we define

$$\begin{aligned} \tilde{S} &= S_{SH} + \lambda \left[\sum_i p_- 1 \right] \\ &= \sum_i p_i \log_2 p_i + \lambda \left[\sum_i p_- 1 \right] \end{aligned}$$

And we say that we are going to have a message that is N letters long and since the message is being transmitted in bits we'll say $N = 2^n$ where n is the number of bits used to transmit the message of length N. Anyway, maximize $\tilde{S}$ for all variables:

$$\begin{aligned} \frac{\partial}{\partial p_j} \tilde{S} &= \frac{\partial}{\partial p_i} [p_i \log_2 p_i] + \sum_{j \neq i} 0 + \frac{\partial}{\partial p_i} \lambda p_i + \sum_{j \neq i} 0 - 0 \\ &= \frac{\ln p_i}{\ln 2} = \frac{1}{\ln 2} + \lambda \\ &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\ln p_i}{\ln 2} = \frac{1}{\ln 2} + \lambda = 0 \\ \ln p_i + 1 + \lambda \ln 2 = 0 \\ \ln p_i = -1 - \lambda \ln 2 \\ p_i = \exp(-1 - \lambda \ln 2) \end{aligned}$$

But $\exp(-1 - \lambda \ln 2)$ is a constant $\forall i$. For now let's say $\exp(-1 - \lambda \ln 2) = p$.

We didn't maximize by λ yet:

$$\frac{\partial}{\partial \lambda} \tilde{S} = \sum_i^N p_i - 1 = 0$$

Which is just the original condition. But now we know that $p_i = p$ so we can say

$$
\begin{aligned}
\sum_{i=1}^{N} p_i &= 1 \\
\sum_{i=1}^{N} p &= 1 \\
p\sum_{i=1}^{N} 1 &= 1 \\
pN &= 1 \\
p_i &= \frac{1}{N} \quad \forall\, i
\end{aligned}
$$

This was very expected. Using this maximizing probability distribution we find the Shannon entropy of on transmission:

$$
\begin{aligned}
S &= -k_s \sum_i p_i \ln p_i \\
&= -\frac{1}{\ln 2} \sum_{i=1}^{N} \frac{1}{N} \ln\left(\frac{1}{N}\right) \\
&= \frac{1}{\ln 2}\frac{1}{N} \ln N \sum_{1}^{N} 1 \\
&= \frac{1}{\ln 2}\frac{N}{N} \ln N \\
&= \frac{\ln N}{\ln 2} \\
&= \frac{\ln 2^n}{\ln 2} \\
&= n\frac{\ln 2}{\ln 2} \\
&= \boxed{n}
\end{aligned}
$$

So the maximum Shannon entropy of a binary message of length $N = 2^n$ where $n =$ the number of bits used, is equal to n. That is to say that the maximum is one unit of Shannon entropy per bit.

4.3 Problem 15 d)

Find the compression scheme that saturates to $S = n$.

From **Part a)** $S = k_s \frac{3}{2} \ln 2$ and since we will be compressing the language into bits we set $k_s = 1/\ln 2$ which leads to

$$S = \frac{3}{2}$$

This says that the average number of bits for each letter is 3/2 or 1.5.

C occurs with the greatest probability so let's insist that it is a single null bit:

$$\boxed{C \to 0}$$

Now the trick to this is that me must be able to distinguish any letter from any combination of letters. For example, say $C \to 0$, $B \to 1$ and $A \to 10$. This is **not** and acceptible compression since there is no distinguishing BC from A. We could expect this since the number of bits per letter would have been $\frac{1}{2} \times 1 + \frac{1}{4} \times 1 + \frac{1}{4} \times 2$.

An acceptible compression would be

$$\boxed{C \to 0 \qquad B \to 11 \qquad A \to 10}$$

Notice A could not just be 1.

Now the average number of bits is $\frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \boxed{\frac{3}{2}}$

, as desired.