Marion and Thornton

Tyler Shendruk

October 1, 2010

1 Marion and Thornton Chapter 7

Hamilton's Principle - Lagrangian and Hamiltonian dynamics.

1.1 Problem 6.4

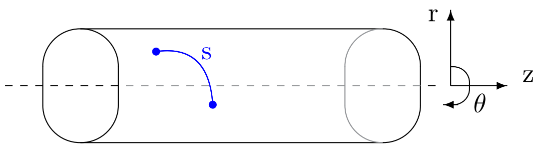

Figure 1: Geodesic on circular cylinder

Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.

Recall that a geodesic is the shortest path along the surface between any two points on that surface. We know that an element of length on the surface of a cylinder is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2. \quad (1)$$

On a given cylinder of radius $r = R$ we can have variation of both θ and z. We can choose either to be our independent variable.

z as independent variable Writing Eq. (1) with z as the independent variable, the total length between points 1 and 2 is

$$\begin{aligned} s &= \int_{z_1}^{z_2} ds = \int_{z_1}^{z_2} \sqrt{dr^2 + r^2 d\theta^2 + dz^2} \\ &= \int_{z_1}^{z_2} \sqrt{0^2 + r^2 \left(\frac{d\theta}{dz}\right)^2 + 1} dz = \int_{z_1}^{z_2} \sqrt{1 + r^2 \theta'^2} dz \\ &= \int_{z_1}^{z_2} f\left(\theta, \theta'; z\right) dz \end{aligned}$$

where we have identified $f(\theta, \theta'; z) = \sqrt{1 + r^2\theta'^2}$ to be the function that we will apply Euler's equation to inorder to get an extremum for the path length s.

Apply Euler's equation to f gives

$$\begin{aligned}
\frac{\partial f}{\partial \theta} - \frac{d}{dz}\frac{\partial f}{\partial \theta'} &= 0 \\
\frac{\partial\sqrt{1+r^2\theta'^2}}{\partial \theta} - \frac{d}{dz}\frac{\partial\sqrt{1+r^2\theta'^2}}{\partial \theta'} &= 0 \\
0 - \frac{d}{dz}\left(\frac{r^2\theta'}{\sqrt{1+r^2\theta'^2}}\right) &= 0 \\
\frac{d}{dz}\left(\frac{r^2\theta'}{\sqrt{1+r^2\theta'^2}}\right) &= 0 \qquad (2)
\end{aligned}$$

Therefore, since r is constant θ' is constant with respect to z which indicates

$$\boxed{\theta = c_1 z + c_2}, \qquad (3)$$

the equation for a helix.

θ as independent variable If instead we write Eq. (1) with θ as the independent variable, the length becomes

$$\begin{aligned}
s &= \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta \\
&= \int_{\theta_1}^{\theta_2} \sqrt{r^2+z'^2}\, d\theta = \int_{\theta_1}^{\theta_2} f\left(z, z'; \theta\right) d\theta.
\end{aligned}$$

Euler's equation now gives

$$\begin{aligned}
\frac{\partial f}{\partial z} - \frac{d}{d\theta}\frac{\partial f}{\partial z'} &= 0 \\
\frac{\partial\sqrt{r^2+z'^2}}{\partial z} - \frac{d}{d\theta}\frac{\partial\sqrt{r^2+z'^2}}{\partial z'} &= 0 \\
0 - \frac{d}{d\theta}\left(\frac{z'}{\sqrt{r^2+z'^2}}\right) &= 0 \qquad (4)
\end{aligned}$$

Now z' is constant so

$$\begin{aligned}
z &= c_3\theta + c_4 \\
\theta &= \frac{1}{c_3} z - c_4
\end{aligned}$$

$$\boxed{\theta = c_1 z + c_2} \qquad (5)$$

which is exactly the same as Eq. (3) .

Second Form Of course, one can use Euler's second form for either of these two variables

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} \qquad \text{for } \left(\frac{\partial f}{\partial x} = 0\right) \qquad (6)$$

where

$$\begin{cases} f = \sqrt{r^2 + z'^2} & \text{and } x \to r \quad \text{for } y \to z \\ f = \sqrt{1 + r^2\theta'^2} & \text{and } x \to r \quad \text{for } y \to \theta. \end{cases}$$

Let's choose z. This choice gives

$$\begin{aligned} c_0 &= f - y'\frac{\partial f}{\partial y'} \\ &= \sqrt{r^2 + z'^2} - z'\frac{\partial}{\partial z'}\sqrt{r^2 + z'^2} \\ &= \sqrt{r^2 + z'^2} - z'\frac{z'}{\sqrt{r^2 + z'^2}} \\ c_0\sqrt{r^2 + z'^2} &= r^2 + z'^2 - z'^2 \\ r^2 + z'^2 &= \frac{r^4}{c_0^2} \end{aligned}$$

$$\boxed{z'^2 = \frac{r^4}{c^2} - r^2 = c_1}. \tag{7}$$

This indicates once again z' is a constant and we're at the same conclusion as previously.

And for θ we get

$$\begin{aligned} k_0 &= f - y'\frac{\partial f}{\partial y'} \\ &= \sqrt{1 + r^2\theta'^2} - \theta'\frac{\partial}{\partial \theta'}\sqrt{1 + r^2\theta'^2} \\ &= \sqrt{1 + r^2\theta'^2} - \theta'\frac{\theta' r^2}{\sqrt{1 + r^2\theta'^2}} \\ 1 &= k_0\sqrt{1 + r^2\theta'^2} \\ \frac{1}{k_0^2} &= 1 + r^2\theta'^2 \end{aligned}$$

$$\boxed{\theta'^2 = \frac{1}{r^2 k_0^2} - \frac{1}{r^2} = k_2}. \tag{8}$$

1.2 Problem 7.1

What are the coordinates needed to describe a disk that is rolling on a horizontal plane and is free to rotate about vertical axis of the plane?

The position of the disk can be adequately described by x and y but unlike say a cylinder rolling down an incline the distance gone in either x or y can not be related to the angle that the disk has rolled through (θ in Fig. 2). This is because the disk is free to rotate about z. The angle describing the spinning (called ϕ in the figure) determines the direction in x and y.

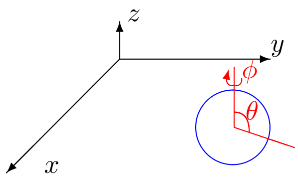

Figure 2: Freely rotating, rolling disk.

If R is the disk radius then the elemental path length is given by

$$ds = Rd\theta = \cos(\phi)\, dx + \sin(\phi)\, dy$$

and by trigonometry in the xy-plane the first of two differential equations (constraints) is

$$\boxed{\tan(\phi) = \frac{dy}{dx}}. \tag{9}$$

In order to get the second differential equation we will square the elemental path along the flat surface and state the radial form and the cartesian form:

$$\begin{aligned} ds^2 &= (Rd\theta)^2 \\ &= dx^2 + dy^2 \end{aligned}$$

Rearrange to get the differential equation

$$\boxed{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = R^2} \tag{10}$$

which is not integrable and since there are no imaginable equations to link the coordinates, the constraints are non-holonomic.

1.3 Problem 7.4

Consider a particle constrained to a plane that moves under force $f_r = -Ar^{\alpha-1}$ which means that the potential is

$$U = -\int \vec{f_r} \cdot d\vec{r} = \int Ar^{\alpha-1} dr = \frac{A}{\alpha} r^\alpha + C.$$

If $U(r=0) = 0$ then the integration constant C drops out. The kinetic energy in cylindrical coordinates is

$$T = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m \left(r\dot{\theta}\right)^2.$$

Combining these gives the Lagrangian

$$\mathcal{L} = T - U = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m \left(r\dot{\theta}\right)^2 - \frac{A}{\alpha} r^\alpha.$$

Lagrange equation for r

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0$$
$$0 + mr\dot{\theta}^2 - Ar^{\alpha-1} - \frac{d}{dt}\left[m\dot{r} + 0 + 0\right] = 0$$
$$mr\dot{\theta}^2 - Ar^{\alpha-1} - m\ddot{r} = 0$$

Lagrange equation for θ

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$
$$0 + 0 + 0 - \frac{d}{dt}\left[0 + mr^2\dot{\theta} + 0\right] = 0$$
$$\frac{d}{dt}\left[L\right] = 0$$

where we've identified $L = mr^2\dot{\theta}$ to be the angular momentum and demonstrated that it is conserved with time.

1.4 Problem 7.5

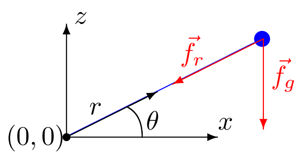

Figure 3: Particle confined to a plane.

Again consider a particle constrained to a plane that moves under force $f_r = -Ar^{\alpha-1}$ but now also under a gravitational force $f_g = mgz$. Now the potential is

$$U = -\int \vec{f}\cdot d\vec{r} = \int Ar^{\alpha-1}dr + \int mgdz = \frac{A}{\alpha}r^\alpha + mgz + C$$
$$= \frac{A}{\alpha}r^\alpha + mgr\sin\theta + C. \tag{11}$$

And again, if $U(r = 0) = 0$ then the integration constant C drops out. The kinetic energy is unchanged from

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(r\dot{\theta}\right)^2. \tag{12}$$

The Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(r\dot{\theta}\right)^2 - \frac{A}{\alpha}r^\alpha - mgr\sin\theta. \tag{13}$$

Lagrange equation for r

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0$$
$$0 + mr\dot{\theta}^2 - Ar^{\alpha-1} - mg\sin\theta - \frac{d}{dt}\left[m\dot{r} + 0 - 0 - 0\right] = 0$$
$$mr\dot{\theta}^2 - Ar^{\alpha-1} - mg\sin\theta - m\ddot{r} = 0$$

$$\boxed{\ddot{r} - r\dot{\theta}^2 + \frac{A}{m}r^{\alpha-1} + g\sin\theta = 0}. \tag{14}$$

Lagrange equation for θ

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= 0 \\
0 + 0 - 0 - mgr\cos\theta - \frac{d}{dt}\left[0 + mr^2\dot{\theta} - 0 - 0\right] &= 0 \\
\frac{dL}{dt} &= -mgr\cos\theta \neq 0. \qquad (15)
\end{aligned}$$

Unlike before the angular momentum (notice it's out of the plane) about the origin is not conserved. Continuing with the Lagrange equation for θ we find

$$\begin{aligned}
\frac{d}{dt}mr^2\dot{\theta} &= -mgr\cos\theta \\
r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\cos\theta &= 0
\end{aligned}$$

$$\boxed{r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\cos\theta = 0}. \tag{16}$$